Regularity theory for the spatially homogeneous Boltzmann equation with cut-off

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Abstract

We develop the regularity theory of the spatially homogeneous Boltzmann equation with cut-off and hard potentials (for instance, hard spheres), by (i) revisiting the L^p-theory to obtain constructive bounds, (ii) establishing propagation of smoothness and singularities, (iii) obtaining estimates about the decay of the singularities of the initial datum. Our proofs are based on a detailed study of the "regularity of the gain operator". An application to the long-time behavior is presented.

Contents

1 Introduction

This paper is devoted to the study of qualitative properties of solutions to the spatially homogeneous Boltzmann equation with cut-off and hard potentials. In this work, we shall obtain new, quantitative bounds on the norms of the solutions in Lebesgue and Sobolev spaces. Before we explain our results and methods in more detail, let us introduce the problem in a precise way.

The spatially homogeneous Boltzmann equation decribes the behavior of a dilute gas, in which the velocity distribution of particles is assumed to be independent on the position; it reads

$$\frac{\partial f}{\partial t} = Q(f,f), \qquad v \in \mathbb{R}^N, \quad t \geq 0,$$

where the unknown $f = f(t,v)$ is a time-dependent probability density on $\mathbb{R}^N$ ($N \geq 2$) and Q is the quadratic Boltzmann collision operator, which we define by the bilinear form

$$Q(g,f) = \int_{\mathbb{R}^N} dv_* \int_{\mathbb{S}^{N-1}} d\sigma B(|v-v_*|, \cos\theta)(g'_* f' - g_* f).$$

Here we have used the shorthands $f' = f(v')$, $g_* = g(v_*)$ and $g'_* = g(v'_*)$, where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v^*}{2} - \frac{|v-v_*|}{2}\sigma$$

stand for the pre-collisional velocities of particles which after collision have velocities v and v_*. Moreover $\theta \in [0,\pi]$ is the deviation angle between $v'-v'_*$ and $v-v_*$, and B is the Boltzmann collision kernel (related to the cross-section $\Sigma(v-v_*,\sigma)$ by the formula $B = \Sigma|v-v_*|$), determined by physics. On physical grounds, it is assumed that $B \geq 0$ and that B is a function of $|v-v_*|$ and $\cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle$.

In this paper we shall be concerned with the case when B is **locally integrable**, an assumption which is usually referred to as *Grad's cut-off assumption* (see [14]). The main case of application is that of hard sphere interaction, where (up to a normalization constant)

$$B(v-v_*,\sigma) = |v-v_*|. \tag{1.1}$$

We shall study more general kernels than just (1.1), but, in order to limit the complexity of statements, we shall assume that B takes the simple product form

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)\, b(\cos\theta), \qquad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle. \tag{1.2}$$

Let us state our assumptions in this context:

- Grad's cut-off assumption, which takes here the simple form

$$\int_0^\pi b(\cos\theta)\sin^{N-2}\theta\,d\theta \;< +\infty. \tag{1.3}$$

It is customary in physics and in mathematics to study the case when Φ behaves like a power law $|v-v_*|^\gamma$, and one traditionally separates between hard potentials ($\gamma > 0$), Maxwellian potentials ($\gamma = 0$), and soft potentials ($\gamma < 0$). Here we shall concentrate on **hard potentials**, and more precisely we shall assume that Φ behaves like a positive power of $|v-v_*|$, in the following sense:

- There exists a $\gamma \in (0,2)$ such that

$$\Phi(0) = 0 \quad \text{and} \quad C_\Phi \equiv \|\Phi\|_{C^{0,\gamma}(\mathbb{R}_+)} < +\infty. \tag{1.4}$$

Here $C^{0,\gamma}(\mathbb{R}_+)$ is the γ-Hölder space on $\mathbb{R}_+$, i.e

$$\|\Phi\|_{C^{0,\gamma}(\mathbb{R}_+)} = \sup_{r,s\in\mathbb{R}_+,\ r\neq s} \frac{|\Phi(r)-\Phi(s)|}{|r-s|^\gamma};$$

- In addition to (1.3) we shall assume a polynomial control on the convergence of the angular integral:

$$\left| \int_0^\pi b(\cos\theta)\sin^{N-2}\theta\,d\theta - \int_\varepsilon^{\pi-\varepsilon} b(\cos\theta)\sin^{N-2}\theta\,d\theta \right| \le C_b\varepsilon^\delta, \quad \text{for some} \;\; \delta > 0. \tag{1.5}$$

Remark: The goal of this assumption is to simplify the computations and bounds which will be derived. Of course, the L^1 integrability of the angular cross-section implies that the left-hand side in (1.5) goes to 0 as $\varepsilon \to 0$, and almost all the results in the present paper remain true under this sole assumption.

- Finally, we shall impose a lower bound on the kernel B, in the form

$$\int_{\mathbb{S}^{N-1}} B(v-v_*,\sigma)\,d\sigma \ge K_B|v-v_*|^\gamma \qquad (K_B > 0, \quad \gamma > 0). \tag{1.6}$$

For a kernel in product form, as in (1.2), this assumption means that b is not identically (almost everywhere) zero and Φ satisfies

$$\Phi(|z|) \ge K_\Phi |z|^\gamma \qquad \forall z \in \mathbb{R}^N \tag{1.7}$$

for some $K_\Phi > 0$.

Remarks: 1. Our assumptions imply that Φ is bounded from above and below by constant multiples of $|v - v_*|^\gamma$. In fact, to establish the subsequent L^p estimates on Q^+, it is sufficient to treat this case: since the gain operator behaves in a monotone way with respect to the collision kernel, the general estimates follow immediately.

2. It would also be immediate to generalize our results to the case in which B is a finite sum of products of the form (1.2), but much more tedious to do the same for a general B, even if no conceptual difficulty should arise.

The Cauchy problem for hard and Maxwellian potentials is by now fairly well understood (see for example Carleman [9, 10], Arkeryd [3], Mischler and Wennberg [20], Bobylev [7]), while soft potentials still remain more mysterious (see Arkeryd [4], Goudon [13], Villani [27] for partial results).

For hard potentials with $0 < \gamma < 2$, the following results are known:

- **Existence and uniqueness** of a solution as soon as the initial datum f_0 satisfies

$$\int_{\mathbb{R}^N} f_0(v) \left(1 + |v|^2\right) \, dv \ < +\infty. \tag{1.8}$$

  This uniqueness statement in fact holds in the class of solutions with nonincreasing kinetic energy, and the solution satisfies the conservation laws

$$\forall t \geq 0, \quad \int_{\mathbb{R}^N} f(t,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \int_{\mathbb{R}^N} f_0(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv.$$

  This strong uniqueness result is due to Mischler and Wennberg [20]. We note that spurious solutions with increasing kinetic energy can be constructed, see [33].

- **Boltzmann's H-theorem**: let $H(f) = \int_{\mathbb{R}^N} f \log f \, dv$, then $\frac{d}{dt} H(f(t,\cdot)) \leq 0$. In particular, if $H(f_0) < +\infty$, then

$$\forall t \geq 0, \qquad H(f(t,\cdot)) \leq H(f_0).$$

- **Moment bounds** (Povzner [21], Desvillettes [11], Wennberg [30, 32], Mischler and Wennberg [20]): if f_0 satisfies (1.8), then

$$\forall s \geq 2, \quad \forall t_0 > 0, \qquad \sup_{t \geq t_0} \int_{\mathbb{R}^N} f(t,v)(1 + |v|^s) \, dv \ < +\infty.$$

  In words, all moments are bounded for positive times, uniformly as t goes to infinity. This effect has been studied at length in the literature, and is strongly linked to the behavior of the collision kernel as $|v - v_*| \to +\infty$. Some explicit bounds are available [11, 32].

- **Positivity estimates** (Carleman [9], Pulvirenti and Wennberg [22]): without further assumptions, it is known that

$$\forall t_0 > 0,\ \exists K_0 > 0,\ \exists A_0 > 0; \qquad t \geq t_0 \implies \forall v \in \mathbb{R}^N, \quad f(t,v) \geq K_0 e^{-A_0|v|^2}.$$

  This means that there is immediate appearance of a Maxwellian lower bound (the particles immediately fill up the whole velocity space). Again the bounds here are explicit.

- L^p **bounds**: L^p estimates ($p > 1$) have been obtained by several authors: Carleman [9, 10] and Arkeryd [5] for $p = +\infty$, Gustafsson [15, 16] for $1 < p < +\infty$. The bounds given by Carleman and Arkeryd are constructive, while this does not seem to be the case for Gustafsson's one, obtained by an intricate nonlinear interpolation procedure.

Our goal in this work is to complete the picture, while staying in the framework of hard potentials with cut-off, by

- revisiting the L^p theory ($1 < p < +\infty$) and obtain **quantitative estimates** together with improved results (holding true under physically relevant assumptions);
- study in detail the phenomena of **propagation of smoothness** and **propagation of singularities**, which are certainly the main physical consequences of Grad's cut-off assumption.

Unlike Gustafsson's proof, our method does not use the L^∞ theory, nor nonlinear interpolation; it is entirely based on the important property of "regularity of the gain operator", namely the fact that the positive part of the Boltzmann collision operator

$$Q^+(g,f) = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} B\big(|v - v_*|, \cos\theta\big) g'_* \, f' \, d\sigma \, dv_*$$

has a regularizing effect. This phenomenon was discovered by Lions [17, 18], and later studied by Wennberg [31], Bouchut and Desvillettes [8], Lu [19]. On one hand we shall use some of the results in [8], but on the other hand we shall also need some fine versions of the regularization property which do not appear in the above-mentioned references, and this is why we shall devote a whole section to the study of this regularization effect. This part should be of independent interest for researchers in the field, since the Q^+ regularity is at the basis of the study of propagation of regularity for the Boltzmann equation in general, including the full, spatially inhomogeneous Boltzmann equation. Wennberg's work [31] will be the starting point of our investigation.

Since the pioneering papers [17, 18] it was known that the Q^+ regularity was useful for smoothness issues; we shall show here that it is also very powerful for establishing L^p bounds, as was first suggested in Toscani and Villani [26]. In this reference, the case of smoothed soft potentials was considered; here we shall adapt the strategy to the case of hard potentials, which will turn out to be much more technical. Our subsequent study of propagation of smoothness will use these L^p bounds as a starting point, in the case $p = 2$.

Interpolation will play an important role in our estimates, but it will only be linear interpolation, applied to the bilinear Boltzmann operator with one frozen argument (typically, $f \mapsto Q(g, f)$).

Our main results can be summarized as follow: under assumptions (1.3), (1.4), (1.5), and (1.6)

- if the initial datum lies in L^p, then the solution is bounded in L^p, uniformly in time;
- if the initial datum is smooth (say in some Sobolev space), then the solution is smooth, uniformly in time;
- if the initial datum is not smooth, then the solution is not smooth either. However, it can be decomposed into the sum of a smooth part (with arbitrary high degree of smoothness) and a nonsmooth part whose amplitude decays exponentially fast.

All this will be quantified and stated precisely in sections 4 and 5. The L^p propagation result is an improvement of already known results, in the sense that we do not need extra L^p moment condition on the initial datum; the other results are new. As an application, we shall establish some new estimates on the rate of convergence to thermodynamical equilibrium as time goes to infinity. Although these estimates are obtained as a consequence of our regularity study, they will hold true even for nonsmooth solutions.

The plan of the present paper is as follows. First, in section 2, we give some simple estimates on the collision operator in various functional spaces. These estimates will be obtained by simple duality arguments; some of them were essentially well-known even if maybe not in the particular form which we give. Then in section 3 we begin our fine study of the regularity of Q^+. It is only in section 4 that we start looking at *solutions* of the Boltzmann equation; in this section we show that if the initial datum lies in L^p ($1 < p < +\infty$) then the solution is bounded in L^p uniformly in time (besides we prove that a phenomenon of "appearance of L^p moments" occurs, like in the case $p = 1$). In section 5, the main result is a decomposition theorem of the solution into the sum of a smooth part (having arbitrary high degree of smoothness) and a nonsmooth part whose amplitude decays exponentially fast. As a preliminary we shall also prove propagation of smoothness, and thus rather precisely tackle the phenomena of propagation of singularities together with exponential decay. Finally, in section 6 we give an application to the study of long-time behavior of the solution: the decomposition theorem allows one to apply estimates for very smooth solution obtained by the second author in [28], in order to prove rapid convergence to global equilibrium.

The whole paper is essentially self-contained, apart from a few simple auxiliary estimates for which precise references will be given, and from known existence and uniqueness results, which we here admit. Some facts from linear interpolation theory and harmonic analysis, used within the proofs, will be recalled in an appendix.

2 Preliminary estimates on the collision operator

Let us first introduce the functional spaces which will be used in the sequel. Throughout the paper we shall use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ and we shall denote by "cst" various constants which do not depend on the collision kernel B. Whenever multi-indices are needed we shall use the common notations $x^\nu = x_1^{\nu_1} \cdots x_N^{\nu_N}$, $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_N^{\nu_N}$, where $\partial_i = \partial / \partial_{x_i}$, and $\binom{\nu}{\mu} = \binom{\nu_1}{\mu_1} \cdots \binom{\nu_N}{\mu_N}$. We shall use weighted Lebesgue spaces L^p_k ($p \geq 1$, $k \in \mathbb{R}$) defined by the norm

$$\|f\|_{L^p_k(\mathbb{R}^N)} = \left(\int |f(v)|^p \langle v \rangle^{pk} dv \right)^{1/p}$$

with the convention

$$\|f\|_{L^\infty_k(\mathbb{R}^N)} = \sup_{v \in \mathbb{R}^N} \left[|f(v)| \langle v \rangle^k \right].$$

We shall also use weighted Sobolev spaces $W^{s,p}_k(\mathbb{R}^N)$; when $s \in \mathbb{N}$ they are defined by the norm

$$\|f\|_{W^{s,p}_k(\mathbb{R}^N)} = \left(\sum_{|\nu| \leq s} \|\partial^\nu f\|^p_{L^p_k} \right)^{1/p}.$$

Then the definition is extended to positive (real) values of s by interpolation. In particular, we shall denote $W^{s,2}_k = H^s_k$; note that this is a Hilbert space.

We shall make frequent use of the translation operators τ_h defined by

$$\forall \ v \ \in \ \mathbb{R}^N, \quad \tau_h f(v) = f(v - h).$$

The translation operation does not leave the weighted norms invariant. Instead, we have the following estimates:

$$\|\tau_h f\|_{L^p_{k_1+k_2}} \leq \langle h \rangle^{|k_2|} \, \|f\|_{L^p_{k_1}}.$$

Finally, we introduce the H functional:

$$H(f) = \int f \log f.$$

For nonnegative functions in L^1_2, $H(f)$ is finite if and only if f belongs to the Orlicz space $L \log L$ defined by the convex function $\phi(X) = (1 + |X|) \log(1 + |X|)$.

2.1 Some convolution-like inequalities on Q^+

In this subsection we prove some estimates on Q^+ in Lebesgue and Sobolev spaces. In the case of Lebesgue spaces, they are essentially contained in Gustafsson [15, 16]; but our method, based on duality, provides somewhat simpler proofs.

We shall establish two different types of estimates: for the bilinear Boltzmann collision operator on one hand, and for the quadratic operator on the other hand. To establish the bilinear estimates, we shall impose an additional assumption on the angular kernel: *no frontal collision should occur*, i.e. $b(\cos\theta)$ should vanish for θ close to π:

$$\exists\, \theta_b > 0 \,; \quad \text{supp } \, b\,(\cos\theta) \subset \{\theta \,/\, \ 0 \le \theta \le \pi - \theta_b\} \tag{2.1}$$

This additional assumption will not be needed, on the other hand, for the quadratic estimates, i.e. the estimates on $Q^+(f,f)$. Indeed, $Q^+(f,g) = \tilde{Q}^+(g,f)$ if $\tilde{Q}^+$ is a Boltzmann gain operator associated with the kernel $\tilde{b}(\cos\theta) = b(\cos(\pi-\theta))$. In particular, $b(\cos\theta)$ and $[b(cos\theta) + b(\cos(\pi-\theta))]1_{\cos\theta\ge 0}$ define the same quadratic operator Q^+, and the latter satisfies (2.1) automatically. We note that $Q^+(g,f)$ and $Q^+(f,g)$ will not necessarily satisfy the same estimates, since assumption (2.1) is not symmetric. To exchange the roles of f and g, we will therefore be led to introduce the assumption that no grazing collision should occur, i.e.

$$\exists\, \theta_b > 0 \,; \quad \text{supp } \, b\,(\cos\theta) \subset \{\theta \,/\, \ \theta_b \le \theta \le \pi\}\,. \tag{2.2}$$

Theorem 2.1. *Let $k, \eta \in \mathbb{R}$, $s \in \mathbb{R}_+$, $p \in [1,+\infty]$, and let B be a collision kernel of the form* (1.2)*, satisfying the assumption* (2.1)*. Then, we have the estimates*

$$\left\| Q^+(g,f) \right\|_{L^p_\eta(\mathbb{R}^N)} \le C_{k,\eta,p}(B)\, \|g\|_{L^1_{|k+\eta|+|\eta|}(\mathbb{R}^N)}\, \|f\|_{L^p_{k+\eta}(\mathbb{R}^N)}\,, \tag{2.3}$$

$$\left\| Q^+(g,f) \right\|_{W^{s,p}_\eta(\mathbb{R}^N)} \le C_{k,\eta,p}(B)\, \|g\|_{W^{\lceil s\rceil,1}_{|k+\eta|+|\eta|}(\mathbb{R}^N)}\, \|f\|_{W^{s,p}_{k+\eta}(\mathbb{R}^N)}\,, \tag{2.4}$$

where $C_{k,\eta,p}(B) = \text{cst} \ (\sin(\theta_b/2))^{\min(\eta,0)-2/p'} \, \|b\|_{L^1(\mathbb{S}^{N-1})}\, \|\Phi\|_{L^\infty_{-k}}$*. If on the other hand assumption* (2.2) *is replaced by assumption* (2.2)*, then the same estimates hold with $Q^+(g,f)$ replaced by $Q^+(f,g)$.*

Corollary 2.2. *Let $k, \eta \in \mathbb{R}$, $p \in [1,+\infty]$, and let B be a collision kernel of the form* (1.2)*. Then we have the estimates*

$$\left\| Q^+(f,f) \right\|_{L^p_\eta(\mathbb{R}^N)} \le C_k(B)\, \|f\|_{L^1_{|k+\eta|+|\eta|}(\mathbb{R}^N)}\, \|f\|_{L^p_{k+\eta}(\mathbb{R}^N)}\,,$$

$$\left\| Q^+(f,f) \right\|_{W^{s,p}_\eta(\mathbb{R}^N)} \le C_k(B)\, \|f\|_{W^{\lceil s\rceil,1}_{|k+\eta|+|\eta|}(\mathbb{R}^N)}\, \|f\|_{W^{s,p}_{k+\eta}(\mathbb{R}^N)}\,, \tag{2.5}$$

where $C_k(B) = \text{cst} \ \ \|b\|_{L^1(\mathbb{S}^{N-1})}\, \|\Phi\|_{L^\infty_{-k}}$.

Remarks: 1. Of course, if B satisfies assumption (1.4), then $C_k(B)$ is finite as soon as $k \ge \gamma$.

2. No regularity is needed on the collision kernel here.

3. In the particular case $\eta \geq 0$, it is possible to obtain slightly better weight exponents in Theorem 2.1 and Corollary 2.2. One can indeed use the inequality

$$|v|^2 \leq |v'|^2 + |v'_*|^2$$

to split the weight on the two arguments of Q^+ and get

$$\left\|Q^+(g,f)\right\|_{L^p_\eta} \leq \text{cst} \left\|Q^+(G,F)\right\|_{L^p}$$

where $F(v) = f(v)\langle v\rangle^\eta$ and $G(v) = g(v)\langle v\rangle^\eta$. When $\eta \geq 0$, the conclusion of Theorem 2.1 thus becomes

$$\left\|Q^+(g,f)\right\|_{L^p_\eta(\mathbb{R}^N)} \leq C_{k,\eta,p}(B) \, \|g\|_{L^1_{k+\eta}(\mathbb{R}^N)} \, \|f\|_{L^p_{k+\eta}(\mathbb{R}^N)} \,,$$

and

$$\left\|Q^+(g,f)\right\|_{W^{s,p}_\eta(\mathbb{R}^N)} \leq C_{k,\eta,p}(B) \, \|g\|_{W^{\lceil s\rceil,1}_{k+\eta}(\mathbb{R}^N)} \, \|f\|_{W^{s,p}_{k+\eta}(\mathbb{R}^N)}$$

4. As we said above, the corollary is obtained from the theorem upon replacing $b(\cos\theta)$ by $[b(\cos\theta)+b(-\cos\theta)]1_{0\leq\theta\leq\pi/2}$. We note that in the case of hard-sphere collision kernel, the physically relevant regime is $\cos\theta \leq 0$, so our trick to reduce to $\cos\theta \geq 0$ should just be considered as a mathematical convenience (which could have been avoided by chosing different conventions; however there is some other motivation for our present conventions).

Proof of Theorem 2.1. By duality,

$$\left\|Q^+(g,f)\right\|_{L^p_\eta} = \sup\left\{\int Q^+(g,f)\psi \;\; ; \;\; \|\psi\|_{L^{p'}_{-\eta}} \leq 1\right\}.$$

We apply the well-known pre-postcollisional change of variables, namely $(v, v_*, \sigma) \to (v', v'_*, (v-v_*)/|v-v_*|)$, which has unit Jacobian, to obtain

$$\int_{\mathbb{R}^N} Q^+(g,f)\psi \, dv = \int_{\mathbb{R}^{2N}} dv \, dv_* \, g_* f \left(\int_{\mathbb{S}^{N-1}} B(|v-v_*|,\sigma)\psi(v') \, d\sigma\right)$$

for all $\|\psi\|_{L^{p'}_{-\eta}} \leq 1$. Let us define the linear operator S by

$$S\psi(v) = \int_{\mathbb{S}^{N-1}} B(|v|,\sigma)\psi\left(\frac{v+|v|\sigma}{2}\right) d\sigma.$$

Then

$$\int_{\mathbb{R}^N} Q^+(g,f)\psi \, dv = \int_{\mathbb{R}^N} g(v_*) \left(\int_{\mathbb{R}^N} f(v) \left(\tau_{v_*} S(\tau_{-v_*}\psi)\right)(v) \, dv\right) dv_*. \tag{2.6}$$

We shall study the operator S in weighted L^1 and L^∞ norms. For brevity we denote $v^+ = \left(\frac{v+|v|\sigma}{2}\right)$. By use of the inequality

$$\sin\left(\frac{\theta_b}{2}\right) |v| \le |v^+| \le |v|$$

which is a consequence of (2.1), we find

$$\|S\psi\|_{L^\infty_{-k-\eta}} \le \text{cst } (\sin(\theta_b/2))^{\min(\eta,0)} \, \|b\|_{L^1(\mathbb{S}^{N-1})} \, \|\Phi\|_{L^\infty_{-k}} \, \|\psi\|_{L^\infty_{-\eta}} \, . \tag{2.7}$$

Next, we turn to the L^1 estimate. First,

$$\begin{aligned}\|S\psi\|_{L^1_{-k-\eta}} &= \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \Phi(|v|) \langle v\rangle^{-k-\eta} \, b(\cos\theta) |\psi(v^+)| \, d\sigma \, dv \\ &\le (\sin(\theta_b/2))^{\min(\eta,0)} \, \|\Phi\|_{L^\infty_{-k}} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} b(\cos\theta) \, |\psi(v^+)| \langle v^+\rangle^{-\eta} \, d\sigma \, dv\end{aligned}$$

The change of variable $v \to v^+$ is allowed because b has compact support in $[0, \pi - \theta_b]$, and its Jacobian is $\frac{2^{N-1}}{\cos^2\theta/2}$. By applying it we find

$$\begin{aligned}\|S\psi\|_{L^1_{-k-\eta}} &\le \text{cst } (\sin(\theta_b/2))^{\min(\eta,0)} \, \|\Phi\|_{L^\infty_{-k}} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} b(\cos\theta) \, |\psi(v^+)| \langle v^+\rangle^{-\eta} \, d\sigma \, dv \qquad (2.8) \\ &\le \text{cst} (\sin(\theta_b/2))^{\min(\eta,0)} \, \|\Phi\|_{L^\infty_{-k}} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} b(\cos\theta) |\psi(v^+)| \langle v^+\rangle^{-\eta} \frac{2^{N-1}}{\cos^2\theta/2} \, dv^+ \, d\sigma \\ &\le \text{cst } (\sin(\theta_b/2))^{\min(\eta,0)-2} \, \|b\|_{L^1(\mathbb{S}^{N-1})} \, \|\Phi\|_{L^\infty_{-k}} \, \|\psi\|_{L^1_{-\eta}} \, .\end{aligned}$$

By the Riesz-Thorin interpolation theorem (see Appendix), from inequalities (2.7) and (2.8) we deduce

$$\|S\psi\|_{L^p_{-k-\eta}} \le C_{k,\eta,p'}(B) \;\; \|\psi\|_{L^p_{-\eta}} \, , \quad 1 \le p \le \infty$$

where $C_{k,\eta,p'}(B) = \text{cst } (\sin(\theta_b/2))^{\min(\eta,0)-2/p} \, \|b\|_{L^1(\mathbb{S}^{N-1})} \, \|\Phi\|_{L^\infty_{-k}}$. Plugging this inequality in (2.6), we find

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} Q^+(g,f)\,\psi\,dv\right| &\leq \int_{\mathbb{R}^N} dv_*|g_*| \left(\int_{\mathbb{R}^N} dv|f|\,|\left(\tau_{-v_*}S(\tau_{v_*}\psi)\right)(v)|\right) \\
&\leq \int_{\mathbb{R}^N} dv_*|g_*|\,\|f\|_{L^p_{k+\eta}}\,\|\tau_{-v_*}S(\tau_{v_*}\psi)\|_{L^{p'}_{-k-\eta}} \\
&\leq \|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^N} |g_*|\langle v_*\rangle^{|k+\eta|}\,\|S(\tau_{v_*}\psi)\|_{L^{p'}_{-k-\eta}}\,dv_* \\
&\leq C_{k,\eta,p}(B)\,\|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^N} |g_*|\langle v_*\rangle^{|k+\eta|}\,\|\tau_{v_*}\psi\|_{L^{p'}_{-\eta}}\,dv_* \\
&\leq C_{k,\eta,p}(B)\,\|f\|_{L^p_{k+\eta}}\,\|\psi\|_{L^{p'}_{-\eta}} \int_{\mathbb{R}^N} |g_*|\langle v_*\rangle^{|k+\eta|+|\eta|}\,dv_* \\
&\leq C_{k,\eta,p}(B)\,\|f\|_{L^p_{k+\eta}} \int_{\mathbb{R}^N} |g_*|\langle v_*\rangle^{|k+\eta|+|\eta|}\,dv_* \\
&\leq C_{k,\eta,p}(B)\,\|f\|_{L^p_{k+\eta}}\,\|g\|_{L^1_{|k+\eta|+|\eta|}}
\end{aligned}$$

This concludes the proof of (2.3).

We now turn to the proof of (2.4). It is based on the formula

$$\nabla Q^\pm(g,f) = Q^\pm(\nabla g, f) + Q^\pm(g, \nabla f) \tag{2.9}$$

which is an easy consequence of the bilinearity and the Galilean invariance property of the Boltzmann operator, namely $\tau_h Q(g,f) = Q(\tau_h g, \tau_h f)$. From (2.9) one can easily deduce a Leibniz formula for derivatives of Q^+ at any order, and equation (2.4) easily follows for any $s \in \mathbb{N}$. Indeed, whenever $s \in \mathbb{N}$ we can apply Theorem 2.1 to each term of the Leibniz formula for $\partial^\nu Q^+(g,f)$ and find

$$\begin{aligned}
\left\|Q^+(g,f)\right\|^p_{W^{s,p}_\eta} &= \sum_{|\nu|\leq s} \left\|\partial^\nu Q^+(g,f)\right\|^p_{L^p_\eta} \\
&= \sum_{|\nu|\leq s}\sum_{\mu\leq\nu}\binom{\nu}{\mu}\left\|Q^+(\partial^\mu g, \partial^{\nu-\mu} f)\right\|^p_{L^p_\eta} \\
&\leq C_{k,\eta,p}(B)\sum_{|\nu|\leq s}\sum_{\mu\leq\nu}\binom{\nu}{\mu}\|\partial^\mu g\|^p_{L^1_{|k+\eta|+|\eta|}}\left\|\partial^{\nu-\mu} f\right\|^p_{L^p_{k+\eta}} \\
&\leq C_{k,\eta,p}(B)\,\|g\|^p_{W^{s,1}_{|k+\eta|+|\eta|}}\,\|f\|^p_{W^{s,p}_{k+\eta}}.
\end{aligned}$$

Then the general case of (2.4) is obtained by use of the Riesz-Thorin interpolation theorem, with respect to the variable f. □

2.2 A lower bound on Q^-

We shall use the following estimates on Q^-.

Proposition 2.3. *Assume that the collision kernel B satisfies* (1.6)*. Then, for all $f \in L^1_2$ with $H(f) < +\infty$, there exists a constant $K(f)$, only depending on a lower bound on $\int f\,dv$, and upper bounds on $\int f|v|^2\,dv$ and $H(f)$, such that*

$$Q^-(f,f) \geq K(f)\, f(v)\, (1+|v|)^\gamma. \tag{2.10}$$

Similarly, if $|v-v_*|^\gamma$ in the right-hand side of (1.6) *is replaced by* $\min(|v-v_*|^\gamma, 1)$*, then the conclusion* (2.10) *should be replaced by*

$$Q^-(f,f) \geq K(f)\, f(v). \tag{2.11}$$

The result is well-known: see for instance [5, lemma 4], or [12, lemma 6].

3 Regularity of the gain operator

It is known since the works of P.-L. Lions [17] that, under adequate assumptions on the collision kernel B, the gain operator $Q^+(g,f)$ acts like a regularizing operator on each of its components when the other one is frozen. In this section we shall establish various versions of this regularizing effect. The results will of course depend on the assumptions imposed on B.

The proof in [17] was very technical; it relied on Fourier integral operators, and the theory of generalized Radon transform (integration over a moving family of hypersurfaces), which was studied in detail by Sogge and Stein at the end of the eighties [23, 24, 25]. Later Wennberg [31] simplified the proof by using the Carleman representation [9] of Q^+, and classical Fourier transform tools. Both authors prove functional inequalities which are roughly speaking of the type

$$\|Q^+(g,f)\|_{H^{(N-1)/2}} \leq C\|f\|_{L^2}\|g\|_{L^1}. \tag{3.1}$$

A slightly different family of inequalities was obtained by much simpler means in independent papers by Bouchut and Desvillettes [8] and Lu [19]: they established functional inequalities of the type

$$\left\|Q^+(f,f)\right\|_{H^{(N-1)/2}} \leq C\, \|f\|^2_{L^2}. \tag{3.2}$$

Four our purposes in the next section, inequalities of type (3.2) will not be sufficient, and we shall need the full strength of inequalities of type (3.1). On the other hand, formulas of the type of (3.2) will be sufficient for our regularity study later in the paper.

The precise variants of (3.1) which will be used in the sequel cannot be found in [31], so we shall re-establish them from scratch. Our proof follows essentially the idea of Wennberg [31], and our main contributions will be to make the constants depend more explicitly on the features of the collision kernel, to extend the results to weighted Sobolev spaces of arbitrary order and arbitrary weight, and to extend the range of admissible collision kernels, allowing a possible deterioration of the exponents of regularization. It would also be possible to adapt the proofs by Sogge and Stein, which are more systematic; but it would be much more tedious to keep track of the constants.

3.1 A splitting of Q^+

We shall first prove the regularity property on the gain operator when the collision kernel is very smooth. Then we shall include the non-smooth part of the kernel, at the price of deteriorating the exponents, by an interpolation procedure with the convolution-like inequalities of section 2. This interpolation is not needed for the proof of propagation of the L^p-bound but will be useful for the study of the propagation of singularity/regularity performed in section 5. This calls for an appropriate splitting of the collision kernel, and therefore of the gain operator.

Let us consider a collision kernel $B = \Phi\, b$ satisfying the general assumptions (1.3), (1.4), (1.5), (1.6). Let $\Theta : \mathbb{R} \to \mathbb{R}_+$ be an even C^∞ function such that supp $\Theta \subset (-1,1)$, and $\int_{\mathbb{R}} \Theta\, dx = 1$ and $\widetilde{\Theta} : \mathbb{R}^N \to \mathbb{R}_+$ be a radial C^∞ function such that supp $\widetilde{\Theta} \subset B(0,1)$ and $\int_{\mathbb{R}^N} \widetilde{\Theta}\, dx = 1$. Introduce the regularizing sequences

$$\begin{cases} \Theta_m(x) = m\,\Theta(mx) & (x \in \mathbb{R}) \\[2mm] \widetilde{\Theta}_n(x) = n^N \widetilde{\Theta}(nx) & (x \in \mathbb{R}^N). \end{cases}$$

We shall use these mollifiers to split the collision kernel into a smooth and a non-smooth part. As a convention, we shall use subscripts S for “smooth” and R for “remainder”. First, we set

$$\Phi_{S,n} = \widetilde{\Theta}_n * (\Phi\; 1_{\mathbb{A}_n})\,, \qquad \Phi_{R,n} = \Phi - \Phi_{S,n},$$

where $\mathbb{A}_n$ stands for the annulus $\mathbb{A}_n = \left\{x \in \mathbb{R}^N\ ;\ \frac{2}{n} \le |x| \le n\right\}$. Similarly, we set

$$b_{S,m} = \Theta_m * (b\; 1_{\mathbb{I}_m})\,, \qquad b_{R,m} = b - b_{S,m},$$

where $\mathbb{I}_m$ stands for the interval $\mathbb{I}_m = \left\{x \in \mathbb{R}\ ;\ -1 + \frac{2}{m} \le |x| \le 1 - \frac{2}{m}\right\}$ (b is understood as a function defined on $\mathbb{R}$ with compact support in $[-1,1]$). Finally, we set

$$Q^+ = Q^+_S + Q^+_R$$

where

$$Q^+_S(g,f) = \int_{\mathbb{R}^N} dv_* \int_{\mathbb{S}^{N-1}} d\sigma\, \Phi_{S,n}(|v-v_*|)\, b_{S,m}(\cos\theta)\, g'_*\, f' \tag{3.3}$$

and

$$Q^+_R = Q^+_{RS} + Q^+_{SR} + Q^+_{RR}$$

with the obvious notations

$$\begin{cases} Q^+_{RS}(g,f) &=& \displaystyle\int_{\mathbb{R}^N} dv_* \int_{\mathbb{S}^{N-1}} d\sigma\, \Phi_{R,n}\, b_{S,m}\, g'_*\, f' \\[4mm] Q^+_{SR}(g,f) &=& \displaystyle\int_{\mathbb{R}^N} dv_* \int_{\mathbb{S}^{N-1}} d\sigma\, \Phi_{S,n}\, b_{R,m}\, g'_*\, f' \\[4mm] Q^+_{RR}(g,f) &=& \displaystyle\int_{\mathbb{R}^N} dv_* \int_{\mathbb{S}^{N-1}} d\sigma\, \Phi_{R,n}\, b_{R,m}\, g'_*\, f'. \end{cases}$$

3.2 Regularity and integrability for smooth collision kernel

In this section we shall prove the regularity property of the gain operator under the assumption that both Φ and b are smooth and compactly supported:

$$\Phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \qquad b \in C_0^\infty(-1,1). \tag{3.4}$$

The assumption (3.4) is obviously satisfied by the smooth part Q_S^+ of the gain operator in the decomposition above. Thus, the results in this section will apply to the mollified operator Q_S^+ in (3.3). Our main result in this section is the

Theorem 3.1. *Let $B(v-v_*,\sigma) = \Phi(|v-v_*|)b(\cos\theta)$ satisfy the assumption* (3.4)*. Then, for all $s \in \mathbb{R}^+$, $\eta \in \mathbb{R}$,*

$$\begin{aligned} \left\|Q^+(g,f)\right\|_{H_\eta^{s+\frac{N-1}{2}}} &\le C_{\mathrm{reg}}(s,B)\,\|g\|_{H^s_\eta}\,\|f\|_{L^1_{2|\eta|}} \\ \left\|Q^+(f,g)\right\|_{H_\eta^{s+\frac{N-1}{2}}} &\le C_{\mathrm{reg}}(s,B)\,\|g\|_{H^s_\eta}\,\|f\|_{L^1_{2|\eta|}} \end{aligned} \tag{3.5}$$

where the constant $C_{\mathrm{reg}}(s,B)$ only depends on s and on the collision kernel, see formulas (3.8) *and* (3.9)*.*

Remark: Of course assumption (3.4) is left invariant under the change $\theta \to \pi/2-\theta$, and therefore the estimates in (3.5) are symmetric under exchange of f and g.

Proof of Theorem 3.1. We shall proceed in four steps, following the method of Wennberg [31]. We shall make use of the elementary lemma A.5 in the Appendix to explicitly control an error term disregarded in [31].

Step 1: The Carleman representation

The idea of Carleman representation (see [9, 10]) is to parametrize Q^+ by the variables v' and v'_* instead of v_* and σ. This change of variable leads to

$$Q^+(g,f) = \int_{\mathbb{R}^N} dv' \int_{E_{v,v'}} dv'_* \,\frac{\Phi(|v-v_*|)\,b(\cos\theta)}{|v-v'|^{N-1}} g'_* f'$$

where $E_{v,v'}$ denotes for the hyperplan orthogonal to $v-v'$ and containing v. Since $|v-v'|/|v-v_*| = \sin(\theta/2)$, we can parametrize the kernel by

$$\frac{\Phi(|v-v_*|)\,b(\cos\theta)}{|v-v'|^{N-1}} = \mathcal{B}(v-v_*, |v-v'|)$$

where

$$\mathcal{B}(v_1,|v_2|) = \frac{\Phi(|v_1|)\,b\left(1-2\left(\frac{|v_2|}{|v_1|}\right)^2\right)}{|v_2|^{N-1}}.$$

The fact that $\mathcal{B}$ is radial according the first variable will not be used in the next step, but will reveal useful in Step 3 where some modified versions of the collision kernel will be needed.

Following [31], we define, for $w \in \mathbb{S}^{N-1}$ and $r, s \in \mathbb{R}$,

$$R_{w,r}g(s) = \int_{w^\perp} \mathcal{B}(z + sw, r)\, g(z + sw)\, dz$$

where $w^\perp$ denotes the hyperplane orthogonal to w going through the origin (this is a weighted Radon transform). Then, for $y \neq 0$ we set

$$\begin{aligned} Tg(y) &= \left[R_{y/|y|,|y|}\right] g(|y|) \\ &= \int_{y+y^\perp} \mathcal{B}(z, y)\, g(z)\, dz. \end{aligned}$$

By an easy computation,

$$Q^+(g, f) = \int_{\mathbb{R}^N} f(v')\, (\tau_{v'} \circ T \circ \tau_{-v'})\, g(v)\, dv'$$

(this is the last formula in [31, section 2]). Thus it becomes clear that regularity estimates on the Radon transform T will result in regularity estimates on Q^+. More precisely, a careful use of Fubini and Jensen theorems leads to

$$\left\|Q^+(g, f)\right\|^2_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{R}^N)} \leq \|f\|_{L^1} \int_{\mathbb{R}^N} |f(v')| \left\|(\tau_{-v'} \circ T \circ \tau_{v'}) g(v)\right\|^2_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{R}^N)} dv',$$

and we see that

$$\left\|Q^+(g, f)\right\|_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{R}^N)} \leq C_{\mathrm{reg}}(s, B)\, \|f\|_{L^1_{2|\eta|}}\, \|g\|_{H^s_\eta(\mathbb{R}^N)},$$

if we define $C_{\mathrm{reg}}(s, B)$ as the best constant in the inequality

$$\|Tg\|_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{R}^N)} \leq C\, \|g\|_{H^s_\eta(\mathbb{R}^N)}. \tag{3.6}$$

Step 2: Estimates of radial derivatives of T

We now start to establish (3.6). As we shall see in the next step, it suffices to study the regularity with respect to the modulus of the relative velocity variable, because the angular derivatives can be controlled by the radial ones. We shall work in spherical coordinates and write $Tg(rw) = R_{w,r}g(r)$ ($r > 0$, $w \in \mathbb{S}^{N-1}$). We introduce the "radial Fourier transform", $\mathcal{RF}$, and the Fourier transform in $\mathbb{R}^N$, $\mathcal{F}$, by the formulas

$$\mathcal{RF}f(\rho w) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dr\, e^{i\rho r} f(rw),$$

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} dv\, e^{iv\cdot\xi} f(v).$$

In particular,

$$\mathcal{RF}\left[\langle r\rangle^\eta Tg\right](\rho w) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dr\, e^{i\rho r} \langle r\rangle^\eta \int_{w^\perp} dz\, \mathcal{B}(z+rw, r) g(z+rw).$$

Let $u = z + rw$. By Fubini's theorem and some simple computations,

$$\mathcal{RF}\left[\langle r\rangle^\eta Tg\right](\rho w) = (2\pi)^{\frac{N-1}{2}} \mathcal{F}\big[g(\cdot)\mathcal{B}(\cdot, |(\cdot, w)|)\langle(\cdot, w)\rangle^\eta\big](\rho w).$$

By this we can estimate the $H_\eta^{s+\frac{N-1}{2}}$ norm according to the radial variable. Let

$$\|Tg\|^2_{H_\eta^{s+\frac{N-1}{2}}(\mathbb{S}^{N-1}\times\mathbb{R})} = \int_{\mathbb{S}^{N-1}} dw \int_{\mathbb{R}} d\rho\, \langle\rho\rangle^{2\left(s+\frac{N-1}{2}\right)} \Big|\mathcal{RF}\big[\langle r\rangle^\eta Tg\big](\rho w)\Big|^2$$

$$= (2\pi)^{N-1} \int_{\mathbb{S}^{N-1}} dw \int_{\mathbb{R}} d\rho\, \langle\rho\rangle^{2\left(s+\frac{N-1}{2}\right)} \Big|\mathcal{F}\big[g(\cdot)\mathcal{B}(\cdot, |(\cdot, w)|)\langle(\cdot, w)\rangle^\eta\big](\rho w)\Big|^2.$$

We change variables to get back to Euclidean coordinates, and find

$$\begin{aligned}
&\|Tg\|^2_{H_\eta^{s+\frac{N-1}{2}}(\mathbb{S}^{N-1}\times\mathbb{R})} \\
&= \quad (2\pi)^{N-1} \int_{\mathbb{R}^N} \langle\xi\rangle^{2s+N-1} |\xi|^{-(N-1)} \left|\mathcal{F}\left[g(\cdot)\mathcal{B}\left(\cdot, \left|\left(\cdot, \frac{\xi}{|\xi|}\right)\right|\right) \left\langle(\cdot, \frac{\xi}{|\xi|})\right\rangle^\eta\right](\xi)\right|^2 d\xi.
\end{aligned}$$

Now we cut this expression into two parts: for $|\xi| > 1$, the inequality $|\xi|^2 > 1/2(1 + |\xi|^2)$ implies that the right-hand side is bounded from above by

$$\begin{aligned}
&(8\pi)^{N-1} \int_{|\xi|>1} \langle\xi\rangle^{2s} \left|\mathcal{F}\left[g(\cdot)\mathcal{B}\left(\cdot, \left|\left(\cdot, \frac{\xi}{|\xi|}\right)\right|\right) \left\langle(\cdot, \frac{\xi}{|\xi|})\right\rangle^\eta\right](\xi)\right|^2 \\
&\quad + (2\pi)^{N-1} 2^{s+\frac{N-1}{2}} \left(\int_{\mathbb{B}^N} \frac{d\xi}{|\xi|^{N-1}}\right) \sup_{|\xi|\le 1} \left|\mathcal{F}\left[g(\cdot)\mathcal{B}\left(\cdot, \left|\left(\cdot, \frac{\xi}{|\xi|}\right)\right|\right) \left\langle(\cdot, \frac{\xi}{|\xi|})\right\rangle^\eta\right](\xi)\right|^2,
\end{aligned}$$

where $\mathbb{B}^N$ stands for the ball of radius 1.

Then, on one hand Lemma A.5 implies

$$\begin{aligned}
&\int_{|\xi|>1} \langle\xi\rangle^{2s} \left|\mathcal{F}\left[g(\cdot)\mathcal{B}\left(\cdot, \left|\left(\cdot, \frac{\xi}{|\xi|}\right)\right|\right) \left\langle(\cdot, \frac{\xi}{|\xi|})\right\rangle^\eta\right](\xi)\right|^2 \\
&\qquad\qquad \le \|g\|^2_{H^s_\eta} \left\|\mathcal{B}\left(x, \left|\left(x, \frac{y}{|y|}\right)\right|\right) \frac{\left\langle(x, \frac{y}{|y|})\right\rangle^\eta}{\langle x\rangle^\eta}\right\|^2_{L^\infty_y(H^S_x)}
\end{aligned}$$

where $S = s + \lfloor N/2 \rfloor + 1$. On the other hand, for each $|\xi| \le 1$,

$$\left| \mathcal{F}\left[g(\cdot)\mathcal{B}(\dots) \left\langle (\cdot, \frac{\xi}{|\xi|}) \right\rangle^{\eta} \right](\xi) \right| = \left| \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} g(x)\mathcal{B}(\dots) \left\langle (x, \frac{\xi}{|\xi|}) \right\rangle^{\eta} dx \right|.$$

Hence, by the Cauchy-Schwarz inequality,

$$\left| \mathcal{F}\left[g(\cdot)\mathcal{B}(\dots) \left\langle (\cdot, \frac{\xi}{|\xi|}) \right\rangle^{\eta} \right](\xi) \right| \le \frac{1}{(2\pi)^{N/2}} \|g\|_{L^2_\eta} \sup_{w\in\mathbb{S}^{N-1}} \left\| \mathcal{B}(x, |(x,w)|) \frac{\langle (x,w)\rangle^\eta}{\langle x\rangle^\eta} \right\|_{L^2(\mathbb{R}^N_x)}.$$

Adding up the previous inequalities, we conclude that

$$\|Tg\|_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{S}^{N-1}\times\mathbb{R})} \le \mathrm{cst}(N,s)\ \|g\|_{H^s_\eta} \sup_{w\in\mathbb{S}^{N-1}} \left\| \mathcal{B}(x, |(x,w)|) \frac{\langle (x,w)\rangle^\eta}{\langle x\rangle^\eta} \right\|_{H^S(\mathbb{R}^N_x)}. \tag{3.7}$$

Step 3: Corollary: estimates of the angular derivatives of T

Here we show how to get estimates on the *angular* derivatives of Tg thanks to the estimates on the radial derivatives. We first require the exponent $s + \frac{N-1}{2}$ to be integer, so that the $H^{s+\frac{N-1}{2}}_\eta$ norm can be computed in terms of norms of derivatives. Then,

$$\frac{\partial (Tg)}{\partial y_i}(y) = \sum_j \frac{\partial w_j(y)}{\partial y_i} \frac{\partial}{\partial w_j} R_{w,s} g(s) + \frac{\partial s(y)}{\partial y_i} \left[\frac{\partial}{\partial s} R_{w,r} g(s) \right]_{s=r} + \frac{\partial r(y)}{\partial y_i} \left[\frac{\partial}{\partial r} R_{w,r} g(s) \right]_{s=r}$$

where

$$w(y) = \frac{y}{|y|}, \qquad r(y) = s(y) = |y|,$$

and higher-order variants of this formula can obviously be obtained by differentiating at arbitrary order. Let us assume that supp $(\Phi) \subset [\alpha, +\infty)$ and supp $(b) \subset [\varepsilon, 1]$ ($\alpha > 0$ and $0 < \varepsilon < 1$). Then supp $(\mathcal{B}) \subset [\alpha, +\infty) \times [\varepsilon\alpha, +\infty)$, and one can easily establish that

$$\left| \frac{\partial^\nu w_j(y)}{\partial y^\nu} \right| \le \frac{\mathrm{cst}(N)}{(\alpha\varepsilon)^{|\nu|}}, \qquad \left| \frac{\partial^\nu s(y)}{\partial y^\nu} \right|, \quad \left| \frac{\partial^\nu r(y)}{\partial y^\nu} \right| \le \frac{\mathrm{cst}(N)}{(\alpha\varepsilon)^{|\nu|-1}}$$

in the support of $\mathcal{B}$.

Our second tool is the following property of the Radon transform: it can be rewritten

$$\begin{aligned} R_{w,r} g(s) &= \int_{w^\perp} g(z+sw)\mathcal{B}(z+sw, r)\, dz \\ &= \int_{\mathbb{R}^N} g(u)\mathcal{B}(u,r)\delta(w\cdot u - s)\, du. \end{aligned}$$

where δ is the Dirac mass at 0 on $\mathbb{R}$. Thus

$$\begin{aligned} \frac{\partial}{\partial w_j} R_{w,r} g(s) &= \int_{\mathbb{R}^N} g(u)\mathcal{B}(u,r) u_j\, \delta'(w\cdot u - s)\, du \\ &= -\frac{\partial}{\partial s} \int_{\mathbb{R}^N} g(u)\mathcal{B}(u,r) u_j\, \delta(w\cdot u - s)\, du \\ &= -\frac{\partial}{\partial s} \widetilde{R}_{w,r}(g)(s) \end{aligned}$$

where $\widetilde{R}_{w,r}$ is defined by the new kernel $\mathcal{B}(u,r)u_j$. Thus the angular derivative $\frac{\partial}{\partial w_j} R_{w,r} g(s)$ can be obtained from the estimate (3.7) of Step 2, upon changing the collision kernel by another one, only differing by a factor of u_j. The same holds true for all order derivatives...

To conclude with the regularization property of T, it is enough to notice that the derivatives along r are already taken into account above, and to use the above-mentioned commutation property for the angular derivatives. We conclude that equation (3.6) holds true with

$$\begin{aligned} C_{\mathrm{reg}}(s,B) \;\; = \;\; & \frac{\mathrm{cst}(s,N)}{(\alpha\varepsilon)^{s+\frac{N-1}{2}}} \\ & \sup \left\{ \left\| \mathcal{B}(x,|(x,w)|) x^{\nu} \frac{\langle (x,w) \rangle^{\eta}}{\langle x \rangle^{\eta}} \right\|_{H^{S-|\eta|}(\mathbb{R}^N_x)} ; \; |\nu| \leq s + \frac{N-1}{2}; \quad w \in \mathbb{S}^{N-1} \right\}. \end{aligned} \tag{3.8}$$

This concludes the proof of (3.6) when $s+(N-1)/2$ is an integer. The general case follows by the Riesz-Thorin interpolation theorem again. □

Order of the constant according to the convolution parameters:

The computation of an upper bound on the constant $C_{\mathrm{reg}}(s,B)$ for the collision kernel $\Phi_{S,n} b_{S,m}$ according to the mollifying parameters m and n is tedious but straightforward. One may easily obtain a polynomial bound in the form

$$\begin{aligned} C_{\mathrm{reg}}(s,B) \;\; & \leq \;\; \mathrm{cst}(s,N) m^{as+b} n^{a's+b'} \left\| 1_{\mathbb{I}_m} b \right\|_{L^1(\mathbb{S}^{N-1})} \\ & \leq \;\; \mathrm{cst}(s,N) m^{as+b} n^{a's+b'} \left\| b \right\|_{L^1(\mathbb{S}^{N-1})}. \end{aligned} \tag{3.9}$$

where a, a', b, b' stand for some constant depending only the dimension N and γ.

We conclude this section with the following corollary of Theorem 3.1, which translates the gain of regularity into a gain of integrability.

Corollary 3.2. *Let us consider a collision kernel B satisfying the smoothness assumption* (3.4)*. Then, for all $p \in (1;+\infty)$, $\eta \in \mathbb{R}$, we have*

$$\begin{aligned} \left\| Q^+(g,f) \right\|_{L^q_\eta} \;\; & \leq \;\; C_{\mathrm{int}}(p,\eta,B) \left\| g \right\|_{L^p_\eta} \left\| f \right\|_{L^1_{2|\eta|}} \\ \left\| Q^+(f,g) \right\|_{L^q_\eta} \;\; & \leq \;\; C_{\mathrm{int}}(p,\eta,B) \left\| g \right\|_{L^p_\eta} \left\| f \right\|_{L^1_{2|\eta|}} \end{aligned}$$

where the constant $C_{\mathrm{int}}(p,\eta,B)$ only depends on the collision kernel, p and η, and $q > p$ is given by

$$q = \begin{cases} \dfrac{p}{2 - \frac{1}{N} + p\left(\frac{1}{N} - 1\right)} & \text{if } p \in (1;2] \\[2ex] pN & \text{if } p \in [2;+\infty). \end{cases}$$

Remark: Just as C_{reg}, the constant $C_{\text{int}}(m,n)$ depends on the mollifying parameters in a polynomial way. Note that the constant $C_{\text{int}}(p,\eta,B)$ in Corollary 3.2 does not depend on the weight exponent η anymore in the quadratic case (just as in section 2).

Proof. The proof is almost obvious. When $p=2$, it is a direct consequence of Theorem 3.1 with $s=0$, and the Sobolev injection $H_\eta^{\frac{N-1}{2}} \hookrightarrow L^{2N}_\eta$ (with a constant only depending on N). The general case follows by a Riesz-Thorin interpolation betweeen this estimate and the convolution-like inequalities in Theorem 2.1. $\square$

3.3 Regularity and integrability for nonsmooth collision kernel

In this paragraph we extend the regularity of Q^+ to general nonsmooth kernels. There are at least two strategies for that, which will lead to slightly different results. We shall first give a general result of "gain of integrability/regularity", in a form which is remindful of the classical Povzner inequalities used to study the L^1 moment behavior (besides it will play the same role in the proof of propagation of L^p moments).

• The following inequality will turn out to be the most appropriate for our study of propagation of integrability. We state it only in its quadratic version, the bilinear version would be slightly more intricate but easy to write down as well.

Theorem 3.3. *Let B be a collision kernel satisfying assumptions* (1.3)*,* (1.4) *and* (1.5)*. Then, for all $p>1$, $k>\gamma$ and $\eta \geq -\gamma$, there exist constants C and κ, and $q<p$ (q only depending on p and N), such that for all $\varepsilon>0$, and for all measurable f,*

$$\|Q^+(f,f)\|_{L^p_\eta} \leq C\varepsilon^{-\kappa}\, \|f\|_{L^q_\eta}\|f\|_{L^1_{2|\eta|}} + \varepsilon \|f\|_{L^p_{\gamma+\eta}}\|f\|_{L^1_{k+2\eta_+}}.$$

This estimate expresses a "mixing" property of the Q^+ operator: the dominant norm $L^p_{\gamma+\eta}$ appears with a constant ε as small as desired; and for the rest, we can lower both the Lebesgue exponent and its weight. This property is of course consistent with the compactness properties of Q^+, and in complete contrast with the properties of the loss term Q^-.

Proof of Theorem 3.3. We split Q^+ as $Q^+_S + Q^+_{RS} + Q^+_{SR} + Q^+_{RR}$ and we shall estimate each term separately. From the beginning we assume, without loss of generality, that the angular kernel $b(\cos\theta)$ has support in $[0,\pi/2]$. Remember that the truncation parameters n (for the kinetic part) and m (for the angular part) are implicit in the decomposition of Q^+.

By Corollary 3.2, there exists a constant $C_{\text{int}}(m,n)$, blowing up polynomially as $m\to\infty$, $n\to\infty$, such that

$$\left\|Q^+_S(f,f)\right\|_{L^p_\eta} \leq C_{\text{int}}(m,n)\|f\|_{L^q_\eta}\|f\|_{L^1_{2|\eta|}},$$

for some $q < p$, namely

$$q = \begin{cases} \dfrac{(2N-1)p}{N+(N-1)p} & \text{if } p \in (1;2N] \\ \\ \dfrac{p}{N} & \text{if } p \in [2N;+\infty) \end{cases}$$

(the roles of p and q are exchanged here with respect to Corollary 3.2....)

Next, we shall take advantage of the fact that $b_{R,m}$ has a very small mass (assumptions (1.3) and (1.5)), and write, using Corollary 2.2 with $k = \gamma$,

$$\|Q^+_{RR}(f,f)\|_{L^p_\eta} \le Cm^{-\delta} \|f\|_{L^1_{|\gamma+\eta|+|\eta|}} \|f\|_{L^p_{\gamma+\eta}},$$

for some constant C only depending on C_Φ. A similar estimate holds true for $\|Q^+_{SR}\|_{L^p_\eta}$. Since $\gamma + \eta \ge 0$, we can write $|\gamma+\eta| + |\eta| = \gamma + 2\eta_+$, where $\eta_+ = \max(\eta, 0)$.

It remains to estimate the term Q^+_{RS}. For this we shall consider separately large and small velocities, and write $f = f_r + f_{r^c}$, where

$$\begin{cases} f_r = f\, 1_{\{|v|\le r\}} \\ f_{r^c} = f\, 1_{\{|v|>r\}}. \end{cases}$$

On the one hand, we use Theorem 2.1, and pick a $k > \gamma$, in order to ensure that $\|\Phi_{R,n}\|_{L^\infty_{-k}}$ goes to 0 as $n \to \infty$. Thanks to the Hölder assumption (1.4), one can easily prove

$$\|\Phi_{R,n}\|_{L^\infty_{-k}} \le \text{cst } n^{-(\min(\gamma, k-\gamma))}.$$

It follows

$$\begin{aligned} \left\|Q^+_{RS}(f,f_r)\right\|_{L^p_\eta} &\le C\, \|f\|_{L^1_{|k+\eta|+|\eta|}} \, \|f_r\|^p_{L^p_{k+\eta}} \, \|\Phi_{R,n}\|_{L^\infty_{-k}} \\ &\le C\, \|f\|_{L^1_{|k+\eta|+|\eta|}} \, r^{k-\gamma} \|f\|_{L^p_{\gamma+\eta}} n^{\gamma-k} \\ &\le C \left(\frac{r}{n}\right)^{k-\gamma} \|f\|_{L^1_{k+2\eta_+}} \|f\|_{L^p_{\gamma+\eta}}. \end{aligned}$$

(here $\theta_b = \pi/2$ thanks to the symmetrization).

Remark: This is the only place where we use a regularity estimate on Φ.

On the other hand, the support of $b_{S,m}$ lies a positive distance ($O(1/m)$) away from 0, so (2.2) holds true with $\theta_b = \text{cst}\, m^{-1}$. Thus we can apply Theorem 2.1 with f and g exchanged, to find

$$\left\|Q^+_{RS}(f,f_{r^c})\right\|_{L^p_\eta} \le Cm^\beta \|f_{r^c}\|_{L^1_{|\gamma+\eta|+|\eta|}} \|f\|_{L^p_{\gamma+\eta}}.$$

where $\beta = \max(-\eta, 0) + 2/p'$ and C depends only on C_Φ. Since we assume $\gamma + \eta \ge 0$, this can also be bounded by

$$Cm^\beta r^{\gamma-k} \|f\|_{L^1_{k+\eta+|\eta|}} \|f\|_{L^p_{\gamma+\eta}} = Cm^\beta r^{\gamma-k} \|f\|_{L^1_{k+2\eta_+}} \|f\|_{L^p_{\gamma+\eta}}.$$

To sum up, we have obtained

$$\|Q^+(f,f)\|_{L^p_\eta} \leq C_1(m,n)\|f\|_{L^q_\eta}\|f\|_{L^1_{2|\eta|}} + C\Big[m^{-\delta} + \Big(\frac{r}{n}\Big)^{k-\gamma} + \frac{m^\beta}{r^{k-\gamma}}\Big]\|f\|_{L^1_{k+2\eta_+}}\|f\|_{L^p_{\gamma+\eta}}.$$

The conclusion follows by choosing first m large enough, then r, then n. $\square$

We turn to another similar theorem in which the emphasis is laid on regularity rather than integrability and whose proof is quite similar.

Theorem 3.4. *Let B be a collision kernel satisfying assumptions* (1.3), (1.4) *and* (1.5). *Then, for all $s > 0$, $k > \gamma$ and $\eta \geq -\gamma$, there exist constants C and κ, and $0 \leq s' < s$ ($s' = \max\big(s - \frac{N-1}{2}, 0\big)$ only depending on s and N), such that for all ε, and for all measurable f,*

$$\|Q^+(f,f)\|_{H^s_\eta} \leq C\varepsilon^{-\kappa}\,\|f\|_{H^{s'}_\eta}\|f\|_{L^1_{2|\eta|}} + \varepsilon\|f\|_{H^s_{\gamma+\eta}}\|f\|_{W^{\lceil s\rceil,1}_{k+2\eta_+}}.$$

Proof of theorem 3.4. The proof follows the same path as the previous one. The term Q^+_S is estimated by Theorem 3.1, the terms Q^+_{SR} and Q^+_{RR} are estimated by Theorem 2.1. For the remaining term Q^+_{RS}, we also estimate separately large and small velocities. But this time, the splitting $f = f_r + f_{r^c}$ should be

$$\begin{cases} f_r = f\,\chi_r \\ f_{r^c} = f - f_r. \end{cases}$$

where χ_r is a C^∞ function with bounded derivatives and such that $\chi_r = 1$ on $|v| \leq r$ and $\operatorname{supp}\chi_r \subset B(0, r+1)$. The end of the proof is straightforward. $\square$

Note that there are other possible variants as well....

• The first way to a regularity result for the full kernel is to use the method by Bouchut and Desvillettes in [8]. Hence it is possible to extend Theorem 2.1 in [8] into the following

Theorem 3.5. *Let $B(v - v_*, \sigma) = \Phi(|v - v_*|)\,b(\cos\sigma)$ be a collision kernel such that Φ satisfies the assumption* (1.4) *and b satisfies*

$$\|b\|_{L^2(\mathbb{S}^{N-1})} < +\infty \tag{3.10}$$

in the sense that $\int b(\cos\theta)^2 \sin^{N-1}\theta\, d\theta < +\infty$. Then for all $s \geq 0$ and $\eta \geq 0$,

$$\big\|Q^+(g,f)\big\|_{H^{s+\frac{N-1}{2}}_\eta} \leq C_{\mathrm{BD}}\left[\|g\|_{H^s_{\eta+\gamma+1}}\,\|f\|_{H^s_{\eta+\gamma+1}} + \|g\|_{L^1_{\eta+\gamma}}\,\|f\|_{L^1_{\eta+\gamma}}\right]$$

where C_{BD} only depends on N and on $\|b\|_{L^2(\mathbb{S}^{N-1})}$.

Remarks: 1. Of course assumption (3.10) is stronger than (1.3); it is however still reasonable in the context of cut-off hard potentials (in particular for hard spheres, in which b is just a constant).

2. The inequality here is not adapted to our study of integrability, but will be useful for our study of regularity. Moreover, the proof and the constants are simpler than those which led us to Theorem 3.6.

• The second way towards a regularity result for the full kernel is to combine Theorem 2.1 and Theorem 3.1 and make an explicit interpolation. By this one can prove the

Theorem 3.6. *Let us consider a collision kernel B satisfying assumptions* (1.3), (1.4) *and* (1.5). *Then for all $k > \gamma$ and $\eta \geq -\gamma$, there exists $\alpha > 0$, depending only on B, such that for all $s \geq 0$ and $\eta \in \mathbb{R}$*

$$\left\|Q^+(f,f)\right\|_{H^{s+\alpha}_\eta} \leq C\, \|f\|_{W^{\lceil s\rceil,1}_{k+2\eta+}}\, \|f\|_{H^s_{\gamma+\eta}},$$

for some constant C which only depends on s and B.

Proof of Theorem 3.6. Let us take $s \in \mathbb{R}_+$ and $\eta \in \mathbb{R}$. We have the following estimates on the four parts of the decomposition of Q^+ (by symmetrization the angular part of the collision kernel is supposed to be zero for $\theta \geq \pi/2$).

1. For the smooth part, Theorem 3.1 gives

$$\left\|Q^+_S(f,f)\right\|_{H^{s+\frac{N-1}{2}}_\eta(\mathbb{R}^N)} \leq C_1\, \|f\|_{L^1_{2|\eta|}}\, \|f\|_{H^s_\eta(\mathbb{R}^N)}$$

where $C_1 = C_{\rm reg}(m,n)$ blows up polynomially as $m \to \infty$, $n \to \infty$.

2. To control the effect of small deviation angles, we use again Corollary 2.2, and the dependence of the constant on $\|b_{R,n}\|$ to ensure it goes to zero; we obtain as in the proof of Theorem 3.3

$$\left\|Q^+_{SR}(f,f),\, Q^+_{RR}(f,f)\right\|_{H^s_\eta} \leq C_2\, \|f\|_{W^{\lceil s\rceil,1}_{\gamma+2\eta+}}\, \|f\|_{H^s_{\gamma+\eta}}$$

where $C_2 = \mathrm{cst}(N)\ \|b_R\|_{L^1(\mathbb{S}^{N-1})}\, \|\Phi\|_{L^\infty_{-\gamma}}$, which thanks to assumption (1.5) can be bounded from above by $\mathrm{cst}(C_B,N) m^{-\delta}$.

3. To control the effect of singularities of the kinetic kernel and high velocities, we use again Theorem 2.1 and pick a $k > \gamma$. As in the proof of Theorem 3.4, we prove

$$\left\|Q^+_{RS}(f,f)\right\|_{H^s_\eta} \leq C_3\, \|f\|_{W^{\lceil s\rceil,1}_{k+2\eta+}}\, \|f\|_{H^s_{\gamma+\eta}}$$

where $C_3 = C\Big[m^{-\delta} + \left(\frac{r}{n}\right)^{k-\gamma} + \frac{m^\beta}{r^{k-\gamma}}\Big]$, which goes to 0 polynomially according to the parameter m when one set r then n as well-chosen functions of m.

To sum up, we know that for all $m \geq 1$, one can decompose Q^+ as $Q^+ = Q^+_{S,m} + Q^+_{R,m}$ (remember n is now set as a function of m), with the estimates

$$\begin{cases} \left\| Q^+_{S,m}(f,f) \right\|_{H^{s+\frac{N-1}{2}}_\eta} \leq C_1 \, \|f\|_{L^1_{2|\eta|}} \, \|f\|_{H^s_\eta} \\ \\ \left\| Q^+_{R,m}(f,f) \right\|_{H^s_\eta} \leq (C_2 + C_3) \, \|f\|_{W^{\lceil s \rceil,1}_{k+2\eta+}} \, \|f\|_{H^s_{\gamma+\eta}} \end{cases}$$

By applying Theorem A.4 in the Appendix, we can conclude that

$$\left\| Q^+(f,f) \right\|_{H^{s+\alpha}_\eta} \leq C \, \|f\|_{W^{\lceil s \rceil,1}_{k+2\eta+}} \, \|f\|_{H^s_\eta}$$

for some $0 < \alpha < \frac{N-1}{2}$ depending on the exponents of polynomial control for each term. This concludes the proof. ☐

Remark: Some closely related results can be found in Wennberg [31], the goal is however different: in this reference the author searches for sufficient conditions on the collision kernel B, to ensure that the $H^{(N-1)/2}$ bound still holds true. Here on the contrary we allow general collision kernels, but, as a natural price to pay, the regularization which we obtain is in general strictly less than a gain of $(N-1)/2$ derivatives.

4 Propagation of L^p estimates

In this section we are interested in the propagation of L^p integrability of the solutions of Boltzmann's equation and its derivatives. Our proofs will be based on a differential inequality approach. Most of the hard job has been done in the functional study of the previous section, and the proofs will be much less technical now.

The bounds that we establish here will later serve as the first step for our study of propagation of regularity via a semigroup approach.

4.1 Main result

Theorem 4.1. *Let $B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos\theta)$ satisfy assumptions* (1.3), (1.4), (1.5), (1.6), *let $1 < p < +\infty$ and let f_0 be a nonnegative function in $L^1_2 \cap L^p(\mathbb{R}^N)$. Then, the unique solution f of the Boltzmann equation with initial datum f_0 satisfies the estimates*

$$\frac{d \, \|f\|^p_{L^p}}{dt} \leq C_+ \, \|f\|^{p(1-\theta)}_{L^p} - K_- \, \|f\|^p_{L^p_{\gamma/p}} \tag{4.1}$$

for some constants $C_+, K_- > 0$, $\theta \in (0,1)$ which only depend on p, N, B, on upper bounds on $\|f\|_{L^1_2}$ and $H(f)$, and on a lower bound on $\|f\|_{L^1}$.

In particular, there is an explicit constant $C_p(f_0)$, only depending on B, on an upper bound on $\|f_0\|_{L^1_2} + \|f_0\|_{L^p}$, and on a lower bound on $\|f_0\|_{L^1}$, such that

$$\forall t \geq 0, \qquad \|f(t,\cdot)\|_{L^p} \leq C_p(f_0).$$

Moreover, for any $t > 0$ and any $\eta > 0$, we know that $f(t,\cdot) \in L^p_\eta(\mathbb{R}^N)$. More precisely, for any $t_0 > 0$,

$$\sup_{t \geq t_0} \|f(t,\cdot)\|_{L^p_\eta} < +\infty.$$

Once again this bound can be computed in terms of B, an upper bound on $\|f_0\|_{L^1_2} + \|f_0\|_{L^p}$, a lower bound on $\|f_0\|_{L^1}$, and a lower bound on t_0.

Proof of Theorem 4.1. Here we shall just be content with establishing the necessary a priori estimates. The proof of the theorem follows from standard approximation arguments, known results on the unique solvability of the Boltzmann equation, with bounds in, say, weighted L^∞ if the initial datum also satisfies such bounds (see the references indicated in the Introduction).

Let f be a solution to the Boltzmann equation, supposed to be in $C^1(\mathbb{R}_t, L^p)$. Also, since the solution is differentiable in L^p,

$$\frac{1}{p}\frac{d\,\|f\|^p_{L^p}}{dt} = \int f^{p-1} Q^+(f,f)\,dv - \int f^{p-1} Q^-(f,f)\,dv.$$

By Proposition 2.3,

$$-\int f^{p-1} Q^- \,dv \leq -K \int f^p (1+|v|)^\gamma \,dv \leq -K_0\,\|f\|^p_{L^p_{\gamma/p}}. \tag{4.2}$$

On the other hand, by Hölder's inequality,

$$\int f^{p-1} Q^+_S(f,f)\,dv \leq \left[\int f^p\right]^{\frac{p-1}{p}} \left[\int (Q^+_S)^p\right]^{\frac{1}{p}}$$

$$= \|f\|^{p-1}_{L^p} \|Q^+_S(f,f)\|_{L^p}.$$

and

$$\int f^{p-1} Q^+_R(f,f)\,dv = \int \left(f\langle v\rangle^{\gamma/p}\right)^{p-1} \frac{Q^+}{\langle v\rangle^{\frac{\gamma}{p'}}} \leq \left[\int (f\langle v\rangle^{\gamma/p})^p\right]^{\frac{p-1}{p}} \left[\int (Q^+_R \langle v\rangle^{-\gamma/p'})^p\right]^{\frac{1}{p}}$$

$$= \|f\|^{p-1}_{L^p_{\gamma/p}} \|Q^+_R(f,f)\|_{L^p_{-\gamma/p'}}.$$

By using the estimates on Q^+_S and Q^+_R proved in Theorem 3.3 with $\eta = -\gamma/p'$, $k = 2$ and $\varepsilon = K_0/(2\|f\|_{L^1_2})$, we can find a constant C, depending on $\|f\|_{L^1_2}$, such that

$$\int f^{p-1} Q^+(f,f)\,dv \leq C\|f\|_{L^q}\|f\|_{L^1}\|f\|^{p-1}_{L^p} + \varepsilon\|f\|_{L^1_2}\|f\|^p_{L^p_{\gamma/p}}$$

where q is defined by (3.3). Combining this with elementary Lebesgue interpolation and the conservation of mass and energy, we deduce that there exists a $\theta \in (0,1)$, only depending on N and p, and a constant C_0, only depending on N, p, B and $\|f_0\|_{L^1_2}$, such that

$$\int f^{p-1} Q^+(f,f)\, dv \leq C_0 \|f\|_{L^p}^{1-p\theta} \|f\|_{L^p}^{p-1} + \frac{K_0}{2} \|f\|^p_{L^p_{\gamma/p}}$$
$$\leq C_0 \|f\|_{L^p}^{p(1-\theta)} + \frac{K_0}{2} \|f\|^p_{L^p_{\gamma/p}}.$$

This together with (4.2) concludes the proof of the differential inequality (4.1) with $C_+ = C_0$ and $K_+ = K_0/2$.

From this differential inequality we see that the time-derivative of $\|f(t,\cdot)\|^p_{L^p}$ is bounded by a constant, and therefore $f(t,\cdot)$ lies in L^p for all times. Moreover, if $\|f(t,\cdot)\|_{L^p_{\gamma/p}}$ ever becomes greater than $(C_+/K_-)^{\frac{1}{p\theta}}$, it follows from (4.1) that $(d/dt)\|f(t,\cdot)\|_{L^p} \leq 0$. Since $\|f\|_{L^p_{\gamma/p}} \geq \|f\|_{L^p}$, we conclude that that

$$C_p(f_0) := \max \left[\|f_0\|_{L^p} \, ; \, \left(\frac{C_+}{K_-}\right)^{\frac{1}{p\theta}} \right]$$

is a uniform upper bound for $\|f(t,\cdot)\|_{L^p}$.

Next, for all $\eta \geq 0$, a similar argument leads to the a priori differential inequality

$$\frac{d\, \|f\|^p_{L^p_\eta}}{dt} \leq C_+ \, \|f\|^{p(1-\theta)}_{L^p_\eta} - K_- \, \|f\|^p_{L^p_{\eta+\gamma/p}} \, . \tag{4.3}$$

where C_+, K_- now depend on the entropy and on some $\|f\|_{L^1_s}$ norm for s large enough (depending on η). We deduce that $\|f\|_{L^p_\eta}$ norms are propagated, uniformly in time, if the initial datum possesses L^1 moments of high enough order. Let $t_0 > 0$ be arbitrarily small; for $t \geq t_0$, we know that all the quantities $\|f(t,\cdot)\|_{L^1_s}$ are bounded, uniformly in time, for all s, and these inequalities therefore hold true with uniform constants as soon as $t \geq t_0$.

We next turn to the property of moment generation, i.e. the proof that L^p_η norms are automatically bounded for positive times. These results are the analogue of the well-known results of L^1 moment generation for hard potential with cut-off (see for instance [32, Theorem 4.2]). Let $t_0 > 0$ be arbitrarily small. Integrating the inequality (4.1) in time from 0 to t_0, we obtain

$$\int_0^{t_0} \|f(s,\cdot)\|^p_{L^p_{\gamma/p}} \leq \frac{C_+}{K_-} \int_0^{t_0} \|f(s,\cdot)\|^{p-\theta}_{L^p} + \frac{1}{K_-}\Big(\|f_0\|^p_{L^p} - \|f(t_0,\cdot)\|^p_{L^p} \Big),$$

which implies

$$\int_0^{t_0} \|f(s,\cdot)\|^p_{L^p_{\gamma/p}} \, ds < +\infty$$

and thus

$$\forall t_0 > 0, \;\; \exists \, t_1 \in (0,t_0); \quad \|f(t_1,\cdot)\|^p_{L^p_{\gamma/p}} < +\infty.$$

Besides, the estimate (4.3) for $\eta = \gamma/p$ gives the propagation of the $L^p_{\gamma/p}$-norm starting from time $t_1 > 0$. Since for $t \geq t_1$, the L^1_s norms of f are uniformly bounded, the argument can be iterated to prove by induction (integrating in time the weighted inequality (4.3)) that

$$\forall \eta \geq 0, \ \forall t > 0, \quad \|f(t,\cdot)\|_{L^p_\eta} < +\infty.$$

The above argument is slightly formal since we worked with quantities which are not a priori finite. It can however be made rigorous and quantitative in the same manner as in [32]. $\square$

Remark: One could also prove the property of moment generation in L^p directly, without induction, by using the idea of Wennberg [32] of comparison to a Bernoulli differential equation. Using the same estimates on Q^+_R and Q^- as in (4.3), convolution-like inequality (2.3) on Q^+_S, and Hölder inequality, one gets the following

$$\frac{d\,\|f\|^p_{L^p_\eta}}{dt} \leq C_+ \|f\|^p_{L^p_\eta} - \frac{K_-}{C_p(f_0)} \|f\|^{p(1+\lambda)}_{L^p_\eta}$$

where $\lambda = \frac{\gamma}{\eta}$ and $C_p(f_0)$ stands for the uniform bound on the L^p norm of the solution. It gives an explicit bound on the L^p moments of the form

$$\forall t > 0, \quad \|f(t,\cdot)\|_{L^p_\eta} \leq \left[\frac{A}{B\left(1 - e^{-A\lambda t}\right)}\right]^{-\frac{\eta}{\gamma}}$$

where A, B depend on $C_p(f_0)$ and an upper bound on L^1 moment of the solution of high enough order. Notice that these bounds are not optimal: for example, $\|f\|_{L^p_{\gamma/p}}$ has to be integrable as a function of t, as $t \to 0^+$, as can be seen from our a priori differential inequality.

4.2 Generalization: propagation of H^k estimates for $k \in \mathbb{N}$

Here we follow the same strategy on the differentiated equation in order to get uniform bounds in Sobolev spaces H^k for $k \in \mathbb{N}$. This method seems to fail for spaces H^k with k non-integer, because fractional derivatives do not behave "bilinearly" with respect to the collision operator. Moreover we state our results only for "power law" kinetic collision kernels. This restriction is made for convenience, and can probably be relaxed at the price of some more work.

Theorem 4.2. *Let $B(v-v_*,\sigma) = |v-v_*|^\gamma\, b(\cos\theta)$ ($\gamma \in (0,2)$) satisfy assumptions* (1.3), (1.5), (1.6) *and* (3.10), *let $\eta \in \mathbb{R}$, and let f_0 be a nonnegative function in L^1_2. Then the unique solution f of the Boltzmann equation with initial datum f_0 satisfies, for any multi-index ν, the estimate*

$$\frac{d}{dt}\|\partial^\nu f\|^2_{L^2_\eta} \leq C_+ \|\partial^\nu f\|_{L^2_\eta} - K_- \|\partial^\nu f\|^2_{L^2_{\eta+\gamma/2}}$$

for some constants $C_+, K_- > 0$, which depend on p, N, B, on upper bounds on $\|f_0\|_{L^1_2} + H(f_0)$, on a lower bound on $\|f_0\|_{L^1}$ and on $L^2_{\eta+1+\gamma}$ norms on derivatives of f of order strictly less than $|\nu|$.

In particular for any $k \in \mathbb{N}$, there is an explicit constant $C_k(f_0)$, only depending on B, on an upper bound on $\|f_0\|_{L^1_2} + \|f_0\|_{H^k_{k(1+\gamma)}}$, and on a lower bound on $\|f_0\|_{L^1}$, such that

$$\forall t \geq 0, \qquad \|f(t,\cdot)\|_{H^k} \leq C_k(f_0).$$

Moreover, for any $t > 0$ and any $\kappa > 0$, we know that $f(t,\cdot) \in H^k_\kappa(\mathbb{R}^N)$. More precisely, for any $t_0 > 0$,

$$\sup_{t \geq t_0} \|f(t,\cdot)\|_{H^k_\kappa} < +\infty.$$

This bound can be computed in terms of B, an upper bound on $\|f_0\|_{L^1_2} + \|f_0\|_{H^k}$, a lower bound on $\|f_0\|_{L^1}$, and a lower bound on t_0.

Proof of Theorem 4.2. Again we only prove the a priori differential inequality: let us consider a given partial derivative $\partial^\nu f$ of f.

$$\begin{aligned} \frac{1}{2} \frac{d \, \|\partial^\nu f\|^2_{L^2_\eta}}{dt} &= \int \partial^\nu f \, \partial^\nu Q^+(f,f) \, \langle v \rangle^{2\eta} \, dv - \int \partial f \, \partial Q^-(f,f) \, \langle v \rangle^{2\eta} \, dv \\ &= \int \partial^\nu f \, \partial^\nu Q^+ \langle v \rangle^{2\eta} \, dv - \int (\partial^\nu f)^2 A * f \, \langle v \rangle^{2\eta} \, dv \\ &\quad - \sum_{0 < \alpha \leq \nu} \binom{\nu}{\alpha} \int \partial^\nu f \, \partial^{\nu-\alpha} f \, \partial^\alpha (A * f) \, \langle v \rangle^{2\eta} \, dv \end{aligned}$$

where $A(z) = \|b\|_{L^1(\mathbb{S}^N)} \Phi(z) = \mathrm{cst}|z|^\gamma$ here. For the first term we apply the regularity theorem 3.5 : since $(N-1)/2 \geq 1$, it implies

$$\left\| \partial^\nu Q^+(f,f) \right\|_{L^2_\eta} \leq C_{\mathrm{BD}} \left[\|f\|_{L^1_{\eta+\gamma}} \, \|f\|_{L^1_{\eta+\gamma}} + \|f\|_{H^{\nu'}_{\eta+\gamma+1}} \, \|f\|_{H^{\nu'}_{\eta+\gamma+1}} \right]$$

where ν' is a multi-index satisfying $|\nu'| < |\nu|$, and thus

$$\int \partial^\nu f \, \partial^\nu Q^+(f,f) \, \langle v \rangle^{2\eta} \, dv \leq C_1 \, \|\partial^\nu f\|_{L^2_\eta}$$

with C_1 depending on the $L^2_{\eta+\gamma+1}$ norm on derivatives of f of order strictly lower than ν and the $L^1_{\eta+\gamma}$ norm of f.

By Proposition 2.3, the second term is bounded by

$$- \int (\partial f)^2 A * f \, \langle v \rangle^{2\eta} \, dv \leq -K_0 \, \|\partial f\|^2_{L^2_{\eta+\gamma/2}}. \tag{4.4}$$

Finally for the third and last term, we split A in $A_S + A_R$ where for $j \in \mathbb{N}$

$$A_S = \left(\widetilde{\Theta}_j * 1_{|v| \geq 2/j} \right) A, \qquad A_R = A - A_S$$

(notice that here we only need to isolate the singularity at zero relative velocity).
For the smooth part,

$$\|\partial^\alpha (A_S * f)\|_{L^\infty} = \|(\partial^\alpha A_S) * f\|_{L^\infty} \le \|\partial^\alpha A_S\|_{L^\infty_{-(\gamma-1)^+}} \|f\|_{L^1_{(\gamma-1)^+}}$$

($\|\partial^\alpha A_S\|_{L^\infty_{-(\gamma-1)^+}} < +\infty$ since $|\alpha| \ge 1$) and thus

$$\int \partial^\nu f \, \partial^{\nu-\alpha} f \, \partial^\alpha (A_S * f) \, \langle v \rangle^{2\eta} \, dv \le C \, \|f\|_{L^1_{(\gamma-1)^+}} \|\partial^\nu f\|_{L^2_\eta} \left\|\partial^{\nu-\alpha} f\right\|_{L^2_\eta} \le C_2 \, \|\partial^\nu f\|_{L^2_\eta}$$

with C_2 depending on on the L^2_η norm on derivatives of f of order strictly less than $|\nu|$ and the $L^1_{(\gamma-1)^+}$ norm of f.
For the remainder term,

$$\|\partial^\alpha (A_R * f)\|_{L^\infty} = \|A_R * (\partial^\alpha f)\|_{L^\infty} \le \|A_R\|_{L^2} \| \partial^\alpha f\|_{L^2}$$

and thus

$$\int \partial^\nu f \, \partial^{\nu-\alpha} f \, \partial^\alpha (A_R * f) \, dv \le C_3 \, \|\partial^\nu f\|_{L^2}$$

if $\alpha < \nu$, with C_3 depending on the L^2 norm on derivatives of f of order strictly lower than ν and the L^1 norm of f, or

$$\int \partial^\nu f \, \partial^{\nu-\alpha} f \, \partial^\alpha (A_R * f) \, dv \le C_3 \, \|\partial^\nu f\|^2_{L^2}$$

if $\alpha = \nu$, with C_3 depending on the L^2 norm of f. In the second case as C_3 goes to zero when j goes to infinity, the term can be damped by the second one thanks to (4.4). This shows that

$$\frac{1}{2} \frac{d \, \|\partial^\nu f\|^2_{L^2}}{dt} \le C_+ \, \|\partial^\nu f\|_{L^2} - K_- \, \|\partial^\nu f\|^2_{L^2_{\gamma/2}}$$

and the proof is complete.

Then the proof of propagation of H^k norm is made by induction. The proof of moments appearance is made first by propagating the $H^k_{-k(\gamma+1)}$ norm, then using interpolation with the L^1 moments.

□

Remark: To get $W^{k,p}$ bounds when p is different from 2, the strategy above could still apply, although with more complications. The idea would be to prove an a priori differential inequality similar to (4.1) on each derivative. One should use the decomposition $Q^+ = Q^+_S + Q^-_R$. To deal with the regular part one should now use Corollary 3.2 instead of Theorem 3.5 on each term of the Leibniz formula; and to deal with the remainder part one should use estimate (2.5), together with the rough estimate

$$\|f\|_{L^1_\eta} \le C(\varepsilon) \|f\|_{L^p_{\eta+N/p'+\varepsilon}}$$

for $\varepsilon > 0$. Moreover the weight exponent in the assumptions become much higher.

5 Propagation of smoothness and singularity via Duhamel formula

The aim of this section is to study the propagation of smoothness and singularity for the solutions of the Boltzmann equation. Throughout the section, we shall consider a given collision kernel B, satisfying assumptions (1.3), (1.4), (1.5), (1.6), (3.10).

5.1 Preliminary estimates

From now on, explicit computations become rather long and we shall try to be as synthetical as possible; so we will not keep track of exact constants. However, all the proofs remain completely explicit and there would be no conceptual difficulty in extracting exact constants.

Our results in the sequel are based on two kinds of estimates. First, a result of stability in L^1 for the solution of the Boltzmann equation with cut-off and hard potential. Secondly, some smoothness estimates on the Duhamel representation formula.

• The stability result in L^1 which we use is an immediate consequence of the estimates in [30] and in [15]. We do not search here for an optimal version. As in the sequel, we shall use the shorthands $f_t = f(t,\cdot)$.

Lemma 5.1. *Let f, g be two solutions of the Boltzmann equation belonging to $L^1_{2+\gamma} \cap L \log L$. Then there exists a constant $C > 0$, only depending on B, such that for all $0 \le k \le 2$ and $t \ge 0$*

$$\frac{d}{dt} \|f_t - g_t\|_{L^1_k} \le C \|f_t - g_t\|_{L^1_k} \|f_t + g_t\|_{L^1_{k+\gamma}}.$$

In particular, as $\|f_t + g_t\|_{L^1_{k+\gamma}}$ is bounded uniformly with respect to t thanks to the assumption, and we have the stability estimate

$$\|f_t - g_t\|_{L^1_k} \le \|f_0 - g_0\|_{L^1_k} e^{C_{\rm stab} t},$$

where $C_{\rm stab}$ only depends on B, $\|f_0\|_{L^1_{k+\gamma}}$ and $\|g_0\|_{L^1_{k+\gamma}}$.

• Next, we introduce the well-known Duhamel representation formula for the Boltzmann equation,

$$\forall\, t \ge 0,\ v \in \mathbb{R}^N, \quad f(t,v) = f_0(v) e^{-\int_0^t Lf(s,v)\, ds} + \int_0^t Q^+(f,f)(s,v) e^{-\int_s^t Lf(\tau,v)\, d\tau}\, ds \quad (5.1)$$

where Lf stands for $A * f$ and $A(z) = \|b\|_{L^1(\mathbb{S}^N)} \Phi(|z|)$. This formula is well-adapted to the study of smoothness issues because it expresses the solution in terms of the initial datum and the regularizing operators Q^+ and L.

For $s \le t$, we set

$$F(s,t,v) = \int_s^t Lf(\tau,v)\,d\tau, \qquad G(s,t,v) = e^{-F(s,t,v)}.$$

We shall prove several estimates on these functions. We look for uniform (with respect to time) estimates, which leads us to allow a "loss" on the weight exponent.

Proposition 5.2. *Let $\alpha, \beta > 0$ be such that $A \in H^{\alpha}_{-\beta}$. Let $\alpha' = \min(\alpha, (N-1)/2)$, and let $\delta = \beta + \gamma + 1$. Then, there is a constant C_{Duh} such that for all $k, \eta \ge 0$,*

$$\left\| \int_0^t Q^+(f,f)(s,\cdot)\, G(s,t,\cdot)\, ds \right\|_{H^{k+\alpha'}_\eta} \le C_{\mathrm{Duh}} \sup_{0 \le \tau \le t} \|f(\tau,\cdot)\|^{\lceil k+\alpha \rceil + 2}_{H^k_{\eta+\delta}}, \tag{5.2}$$

and

$$\|f_0(\cdot)\, G(0,t,\cdot)\|_{H^k_\eta} \le C_{\mathrm{Duh}}\, e^{-K' t}\, \|f_0(\cdot)\|_{H^k_{\eta+\beta}} \sup_{0 \le \tau \le t} \|f(\tau,\cdot)\|^{\lceil k \rceil}_{H^{k-\alpha'}_\beta}, \tag{5.3}$$

with $0 < K' < K$ where $K > 0$ is the constant in (2.10).

Remark: Under our general assumptions, a possible choice of α, β is $\alpha = \gamma$, $\beta = N/2 + \gamma + \varepsilon$, $\varepsilon > 0$. For $\Phi(|z|) = |z|^\gamma$, it would be possible to take $\alpha = \gamma + N/2 - \varepsilon$, for any $\varepsilon > 0$.

Proof of Proposition 5.2. We start with some preliminary estimates on L, F and G. As a consequence of Cauchy-Schwarz inequality, we find that for all $k \ge 0$,

$$\|Lf\|_{W^{k+\alpha,\infty}_{-\beta}} \le C_1 \|f\|_{H^k_\beta}.$$

It follows that

$$\|F(s,t,\cdot)\|_{W^{k+\alpha,\infty}_{-\beta}} \le C_1 \sqrt{t-s} \left(\int_s^t \|f(\tau,\cdot)\|^2_{H^k_\beta}\, d\tau \right)^{1/2}.$$

Combining this with the estimate (2.11), in the form $Lf \ge K$, we deduce that

$$\begin{aligned} \|G(s,t,\cdot)\|_{W^{k+\alpha,\infty}_{-\beta}} &\le C_1 \sqrt{t-s}\, e^{-K(t-s)} \left(\int_s^t \|f(\tau,\cdot)\|^2_{H^k_\beta}\, d\tau \right)^{\frac{\lceil k+\alpha \rceil}{2}} \\ &\le C_2\, e^{-K'(t-s)} \sup_{s \le \tau \le t} \|f(\tau,\cdot)\|^{\lceil k+\alpha \rceil}_{H^k_\beta} \end{aligned} \tag{5.4}$$

with $0 < K' < K$.

Now we use the following simple lemma to exchange a time integral and a $H^S_P(\mathbb{R}^N_v)$ norm:

Lemma 5.3. *Let $Z(s,v)$ be a function on $\mathbb{R}_+ \times \mathbb{R}^N$ and $S, P \in R$, then for any $\lambda > 0$*

$$\left\| \int_0^t Z(s,\cdot)\, ds \right\|_{H^S_P} \leq \frac{1}{\sqrt{\lambda}} \left(\int_0^t e^{\lambda(t-s)} \, \|Z(s,\cdot)\|^2_{H^S_P} \, ds \right)^{1/2}.$$

This lemma is an immediate consequence of the Cauchy-Schwarz inequality with the weight $e^{\frac{\lambda}{2}(t-s)}$, after passing to Fourier variables. The choice of the exponential function is arbitrary; we used it because it is convenient for the sequel.

As a consequence, we have (recall that $\alpha' = \min(\alpha, (N-1)/2)$)

$$\left\| \int_0^t Q^+(f_s, f_s)\, G(s,t,\cdot)\, ds \right\|_{H^{k+\alpha'}_\eta}$$

$$\leq C \left(\int_0^t e^{K'(t-s)} \left\| Q^+(f_s, f_s)\, G(s,t,\cdot) \right\|^2_{H^{k+\alpha'}_\eta} ds \right)^{1/2}$$

$$\leq C \left(\int_0^t e^{K'(t-s)} \|Q^+(f_s,f_s)\|^2_{H^{k+\alpha'}_{\eta+\beta}} \|G(s,t,\cdot)\|^2_{W^{k+\alpha',\infty}_{-\beta}} \, ds \right)^{1/2}.$$

At this stage we apply Theorem 3.5 and estimate (5.4), to get a bound like

$$C \left[\int_0^t e^{K'(t-s)} \, \|f_s\|^4_{H^k_{\eta+\beta+\gamma+1}} e^{-2K'(t-s)} \left(\sup_{s \leq \tau \leq t} \|f(\tau,\cdot)\|^{\lceil k+\alpha \rceil}_{H^k_\beta} \right)^2 ds \right]^{1/2}$$

$$\leq C \left(\int_0^t e^{-K'(t-s)} ds \right)^{1/2} \sup_{0 \leq s \leq t} \|f(s,\cdot)\|^{\lceil k+\alpha \rceil + 2}_{H^k_{\eta+\beta+\gamma+1}} \leq C \sup_{0 \leq s \leq t} \|f(s,\cdot)\|^{\lceil k+\alpha \rceil + 2}_{H^k_{\eta+\beta+\gamma+1}}.$$

This concludes the proof of (5.2).

The proof of (5.3) is performed in a similar way, using estimate (5.4) with $s = 0$. □

5.2 Propagation of regularity

As soon as we have uniform bounds on L^2 moments, the Duhamel representation (5.1) together with Proposition 5.2 imply some uniform bounds on f in Sobolev spaces, provided that the initial datum itself belong to such a space. With respect to the method used for proving Theorem 4.2, the improvement here is that we are able to treat H^s regularity for any $s \in \mathbb{R}_+$. Here is a precise theorem, definitely not optimal.

Theorem 5.4. *Let $0 \leq f_0 \in L^1_2$ be an initial datum with finite mass and kinetic energy, and let f be the unique solution preserving energy. Then for all $s > 0$ and $\eta \geq \beta$, there exists $w(s) > 0$ (explicitly $w(s) = \delta \lceil s/\alpha' \rceil$) such that*

$$f_0 \in H^s_{\eta+w} \Longrightarrow \sup_{t \geq 0} \|f(t,\cdot)\|_{H^s_\eta} < +\infty.$$

Remark: This theorem is not so strong as the decomposition theorem below, because of the strong moment assumption. It is quite likely that the restriction about w could be relaxed with some more work. A sufficient condition for this moment assumption to be automatically satisfied, is that all the L^1 moments of f_0 be finite. Of course we know that for $t \geq t_0$, this is always the case; but this is a priori not sufficient to conclude. Nevertheless it gives by interpolation the following result: under the same assumptions, as soon as $f_0 \in H^s_\eta$, f_t belongs to H^s_η for any $t > 0$. The constant is explicit, is uniformly bounded for $t > t_0$ for any t_0, and blows-up like an inverse power law of t_0 as $t_0 \to 0^+$.

Proof of Theorem 5.4. Let $n \in \mathbb{N}$ be such that $n\alpha' \geq s$ ($n = \lceil s/\alpha' \rceil$). Let $w(s) = \delta \lceil s/\alpha' \rceil$. The proof is made by an induction comprising n steps, proving successively that f is uniformly bounded in $H^{i\alpha'}_{\eta+\frac{n-i}{n}w}$ for $i = 0, 1, ..., n$. The above-mentioned argument is used in each step.

Let us write the induction. The initialisation for $i = 0$, i.e f uniformly bounded in $L^2_{\eta+w}$ is proved by Theorem 4.1 and the more general equation (4.3). Now let $0 < i \leq n$ and suppose the assumption is satisfied for all $0 \leq j < i$. Then proposition 5.2 implies

$$\|f_0(\cdot)\, G(0,t,\cdot)\|_{H^{i\alpha'}_{\eta+\frac{n-i}{n}w}} \leq C_2\, e^{-K't}\, \|f_0(\cdot)\|_{H^{i\alpha'}_{\eta+\frac{n-i}{n}w+\beta}} \sup_{0\leq\tau\leq t} \|f(\tau,\cdot)\|^{\lceil i\alpha' \rceil}_{H^{(i-1)\alpha'}_\beta}.$$

We know from the previous subsection that

$$\left\| \int_0^t Q^+(f,f)(s,\cdot)\, G(s,t,\cdot)\, ds \right\|_{H^{i\alpha'}_{\eta+\frac{n-i}{n}w}} \leq C_3 \sup_{0\leq\tau\leq t} \|f(\tau,\cdot)\|^{\lceil i\alpha' \rceil+2}_{H^{(i-1)\alpha'}_{\eta+\frac{n-i}{n}w+\delta}}$$

Moreover as $\beta \leq \delta \leq w/n$ and $i \geq 1$,

$$\begin{aligned} \beta &\leq \eta + \frac{n-(i-1)}{n} w \\ \eta + \frac{n-i}{n} w + \delta &\leq \eta + \frac{n-(i-1)}{n} w \end{aligned}$$

and thus, using the induction assumption for $i-1$, f is uniformly bounded in $H^{k\alpha'}_{\eta+\frac{n-i}{n}w}$ and the proof is complete. □

5.3 The decomposition theorem

Here we shall give a precise meaning to the idea that the Boltzmann equation with cut-off propagates both smoothness and singularities, but makes the amplitude of the singular part go to zero as time t go to infinity. To this purpose, we shall look for some iterated versions of the Duhamel representation (5.1).

Theorem 5.5. *Let $0 \leq f_0 \in L^1_2 \cap L^2$ and f be the unique energy-preserving solution of the Botzmann equation with initial datum f_0, and let $s \geq 0$, $q \geq 0$ be arbitrarily large.*

Let $\tau > 0$ be arbitrarily small. Then, for all $t \geq \tau$, f can be written $f^S + f^R$, where f^S is nonnegative, and

$$\begin{cases} \sup_{t\geq\tau} \left\| f_t^S \right\|_{H^s_q \cap L^1_2} < +\infty \\ \\ \forall t \geq \tau,\ \forall k > 0,\ \exists \lambda = \lambda(k) > 0; \quad \left\| f_t^R \right\|_{L^1_k} = O\left(e^{-\lambda t}\right). \end{cases}$$

All the constants in this theorem can be computed in terms of the mass, energy and L^2-norm of f_0, and τ.

Remark: The idea of such a decomposition is reminiscent of Wild sums in the case of Maxwellian molecules. Also partial results in this direction were obtained in Wennberg [31] and Abrahamsson [1]. In these cases the gain of regularity in the second term of the Duhamel formula was iterated just once (or twice in [1] for a gain of integrability), and thus the regularity was limited to $H^{(N-1)/2}$ essentially. For hard potentials the obstacle to iterate the Duhamel formula as in the Maxwellian case is the strong non-linearity of the decomposition. Here we bypass this difficulty by the strategy of starting new flows at each step of the iteration.

Proof of Theorem 5.5. We first note that moment estimates imply bounds in L^1_k for all $k \geq 0$, and therefore the only problems are the gain of regularity for the smooth part and the exponential decrease for the remainder part.

The idea of the proof is a use of the Duhamel formula to decompose the flow associated with the equation into two parts, one of which is more regular than the initial datum, while the amplitude of the other decreases exponentially fast with time. We shall use this repeatedly to progressively increase the smoothness: after a while, we start again a new flow having the smooth part of the previous solution as initial datum. And so on. Of course, each time we start a new flow, we shall depart from the true solution, since the initial datum is not the real solution. However, we can use the stability theorem (Lemma 5.1) to control the error.

The times at which we start the new flows are chosen in such a way that the decay of the non-smooth part (measured by the constant C_{dec}) balances the divergence of the solutions (measured by the constant C_{stab}). The idea is summarized in figure 1. Each node of the tree corresponds to a time where we start a new solution of the Boltzmann equation taking for initial data the "smooth part" of the previous solution. In the aim to achieve the goal of balancing the effect of the divergence of the solutions thanks to the exponential decaying of the first term in the Duhamel formula, it is necessary that the decomposition tree ends precisely at the time t we are looking for a decomposition of the solution. Note that for different t, the functions f_t^S constructed below *do not belong to the same flow*.

Let us implement this idea more precisely. By Theorem 4.1, we have a uniform L^2 bound on the solution f, and for a given $t_0 > 0$, we also know that all the L^2-moments

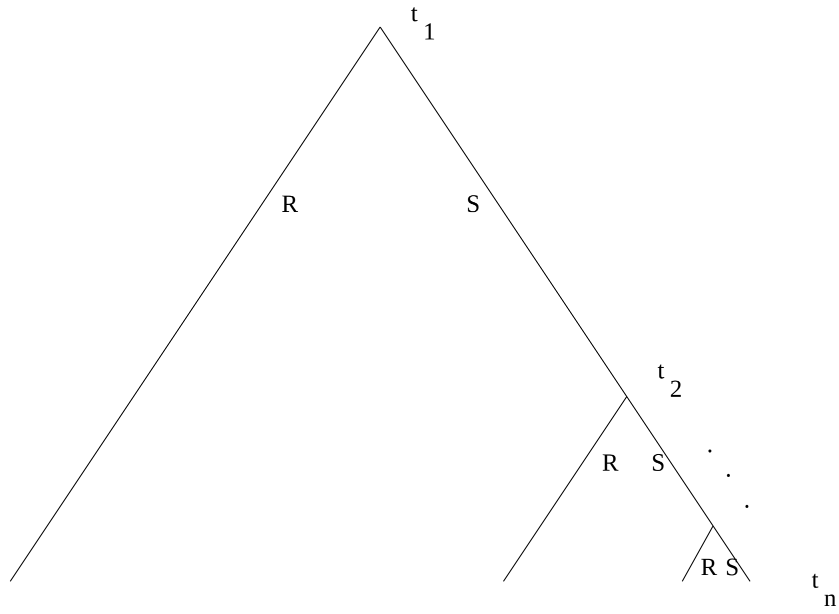

Figure 1: Decomposition of the solution

are uniformly bounded (see subsection 4). Let $n \geq 1$, to be thought of as the number of times we wish to apply the semigroup; we choose n in such a way that $n\alpha' > k$, where k is the degree of smoothness which we are looking for, and α' is the degree of regularization appearing in Proposition 5.2. Let $\tau' \in (0,\tau)$ be arbitrary, say $\tau' = \tau/2$. Let us set $t_n = t \geq \tau$, $t_{-1} = \tau'$, and define inductively (forwards) t_i for $0 \leq i \leq n-1$ by

$$t_i = t_{i-1} + \mu(t_n - t_{i-1})$$

where $\mu \in (0,1)$ satisfies

$$\mu > \frac{C_{\text{stab}}}{C_{\text{stab}} + K'} \tag{5.5}$$

(K' is the constant of exponential decrease in (5.3)). Let us denote $f_0, f_1, ..., f_n$ the solutions constructed as explained above: $f_0 = f$ is the solution we are studying, f_1 is the solution for $t \geq t_0$ of the Boltzmann equation starting from the "initial datum"

$$\int_0^{t_0} Q^+(f_0,f_0)(s,\cdot)\, e^{-\int_s^{t_0} Lf_0}\, ds$$

at time t_0, f_2 is the solution for $t \geq t_1$ of the Boltzmann equation starting from

$$\int_0^{t_1-t_0} Q^+(f_1,f_1)(s,\cdot)\, e^{-\int_s^{t_1-t_0} Lf_1}\, ds$$

at time t_1, etc. More generally, for $2 \leq i \leq n-1$, f_{i+1} is the solution for $t \geq t_i$ of the Boltzmann equation starting at

$$\int_0^{t_i-t_{i-1}} Q^+(f_i,f_i)(s,\cdot)\, e^{-\int_s^{t_i-t_{i-1}} Lf_i}\, ds$$

at time t_i. Of course this sequence is well-defined, since at each node, the "smooth" part of the solution that we take as a new initial data is nonnegative, lies in $L^1_2 \cap L^2$ and has all its L^2-moments bounded.

The n-times iteration of estimate (5.2) together with the theorem of propagation of regularity 5.4 easily implies a bound on the $H^{n\alpha'}$ norm on $f_n(t_n, \cdot)$ which is uniform in $t_n \geq \tau$, and only depends on τ', n, and on the mass, energy and L^2-moments of f. So let us set

$$f^S(t_n, \cdot) \equiv f_n(t_n, \cdot)$$

and

$$f^R(t_n, \cdot) \equiv f(t_n, \cdot) - f^S(t_n, \cdot).$$

This construction can be made for all $t_n \geq \tau$; thus our decomposition is well-defined for all $t \geq \tau$. It remains to prove that f^R is exponentially decaying as $t \to \infty$. For this we write

$$\begin{aligned}
\left\|f^R_t\right\|_{L^1} &= \left\|f^R_{t_n}\right\|_{L^1} \\
&\leq \sum_{i=0}^{n-1} \left\|f^{i+1}_{t_n} - f^i_{t_n}\right\|_{L^1} \\
&\leq \sum_{i=1}^{n} e^{C_{\mathrm{stab}}(t_n - t_i)} \left\|f^{i+1}_{t_i} - f^i_{t_i}\right\| \\
&\leq C \sum_{i=1}^{n} e^{C_{\mathrm{stab}}(t_n - t_i)} e^{-K'(t_i - t_{i-1})} \\
&\leq C \sum_{i=1}^{n} e^{(1-\mu)^{i-1}(t_n - t_0)(C_{\mathrm{stab}}(1-\mu) - K'\mu)} \\
&\leq C \sum_{i=1}^{n} e^{(1-\mu)^{n-1}(t_n - \tau')(C_{\mathrm{stab}}(1-\mu) - K'\mu)} \\
&\leq C n e^{-(t_n - \tau')(1-\mu)^{n-1}(K'\mu - C_{\mathrm{stab}}(1-\mu))}
\end{aligned}$$

which gives the result: if one set

$$0 < C_{\mathrm{dec}} < (1-\mu)^{n-1}\left(K'\mu - C_{\mathrm{stab}}(1-\mu)\right)$$

which is possible thanks to (5.5), we have

$$\left\|f^R_t\right\|_{L^1} \leq C e^{-C_{\mathrm{dec}} t}.$$

On the other hand, f^R_t has all its L^1_k norms bounded, for all k. By elementary interpolation, it follows that all these L^1_k norms are decaying exponentially fast (the same holds true for all L^p_k norms, whenever $p < 2$). $\square$

6 Application to a problem of long-time behavior

Let us now show an application of Theorem 5.5. Here we shall extend a result proven for very smooth solutions, into a result which applies without smoothness assumption.

We start with the following statement, which is an immediate corollary of the main results in [28].

Theorem 6.1. *Let B satisfy assumptions* (1.4), (1.6) *and* (3.10), *together with the stronger lower bound assumption*

$$b(\cos\theta) \geq b_0 > 0. \tag{6.1}$$

Let f_0 be a nonnegative function in $L^1_2(\mathbb{R}^N)$. Without loss of generality, assume that $\int f_0 = 1$, $\int f_0(v)v\,dv = 0$, $\int f_0(v)|v|^2\,dv = N$, and denote by

$$M(v) = \frac{e^{-|v|^2}}{(2\pi)^{N/2}}$$

the associated Maxwellian equilibrium. Let f be an energy-preserving solution of the Boltzmann equation with initial datum f_0, satisfying

$$\forall s,\ \forall k, \qquad C_{s,k} \equiv \sup_{t\geq t_0} \|f_t\|_{H^s_k} < +\infty, \tag{6.2}$$

and

$$\forall t \geq t_0, \qquad f(t,v) \geq K_0 e^{-A_0|v|^{q_0}}.$$

for some time $t_0 > 0$ and some positive constants K_0, A_0, q_0. Then, $\|f_t - M\|_{L^1} = O(t^{-\infty})$, in the sense that for all $\varepsilon > 0$ there exists a constant C_ε, explicitly computable in terms of the above constants, and depending on f only via t_0, K_0, A_0, q_0 and and upper bound on $C_{k,s}$ for k and s large enough, such that

$$\|f_t - M\|_{L^1} \leq C_\varepsilon t^{-1/\varepsilon}. \tag{6.3}$$

Remarks: 1. Assumption (6.1) is satisfied by the hard spheres kernel, and can be considered as satisfactory for hard potentials with cut-off (since they are satisfied by non-cutoff potentials). Kernels like $|v - v_*|^\gamma$ $(0 < \gamma < 2)$ satisfy all the above assumptions.

2. Note that Theorem 6.1 and Theorem 6.2 in the sequel are quantitative, which explains their interest even if exponential convergence to equilibrium has been proven by non-constructive approaches: see Arkeryd [6] for the proof in the L^1 setting, and Wennberg [29] for the extension to the L^p setting.

It is equivalent to require (6.2) or to require uniform bounds in all H^k norms and in all L^1_s norms. Therefore, we see that known results of appearance of moments and Maxwellian lower bound for hard potentials cover all the assumptions needed for this theorem, except the H^k bounds. If we apply the propagation result of Theorem 5.4, we conclude that the conclusion (6.3) holds true as soon as the initial datum f_0 lies in all weighted Sobolev spaces. However, the decomposition theorem of the previous paragraph will lead us to a much stronger conclusion.

Theorem 6.2. *Let f_0 satisfy the same assumptions as in Theorem 6.1 and B satisfy* (1.3), (1.4), (1.5), (1.6), (3.10) *together with* (6.1). *Further assume that $f_0 \in L^2(\mathbb{R}^N)$. Then the conclusion of Theorem 6.1 holds true:*

$$\|f_t - M\|_{L^1} = O(t^{-\infty}).$$

Proof of Theorem 6.2. First of all, let us pick a $t_0 > 0$. We know that the solution f_t satisfies a Maxwellian lower bound and moment estimates, uniformly as $t \geq t_0$.

Let $\varepsilon > 0$ be arbitrary, and let k, s be such that C_ε in Theorem 6.1 only depends on a uniform upper bound on $\|f\|_{H^k_s}$. Let us make the decomposition of Theorem 5.5 with $\tau = 1$ and s. Then we know that

$$f_t = f_t^S + f_t^R, \quad \forall\, t \geq 1.$$

Let $t_1 \geq 1$ be an intermediate time, to be chosen later. Let us introduce $\tilde{f}_t$ the solution of the Boltzmann equation starting from $f_{t_1}^S$ at $t = t_1$, and $\tilde{M}$ the Maxwellian distribution associated with $\tilde{f}_{t_1} = f_{t_1}^S$. Since $f_{t_1}^S$ is bounded in $H^k \cap L^1_s$ by theorem 5.5, $\tilde{f}_t$ is uniformly bounded in $H^k \cap L^1_s$, and has a Maxwellian lower bound for $t \geq t_2 > t_1$ where t_2 can be chosen arbitrarily (so let us say $t_2 = t_1 + 1$). After rescaling space (to reduce to the case where $\tilde{f}$ has unit mass, zero average velocity and unit temperature), Theorem 6.1 implies

$$\|\tilde{f} - \tilde{M}\|_{L^1} = O((t - t_1)^{-\frac{1}{\varepsilon}})$$

with explicit constants which do not depend on t_1 (they only depend on the τ in the decomposition).

Now, thanks to the properties of the decomposition

$$\begin{aligned} \left\|f_t^R\right\|_{L^1_\gamma} &= O\left(e^{-C_{\text{dec}}(t-\tau)}\right) \\ &= O\left(e^{-C_{\text{dec}}(t-1)}\right) = O\left(e^{-C_{\text{dec}} t}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} \left\|M - \tilde{M}\right\|_{L^1} &= O\left(e^{-C_{\text{dec}}(t_1-\tau)}\right) \\ &= O\left(e^{-C_{\text{dec}}(t_1-1)}\right) = O\left(e^{-C_{\text{dec}} t_1}\right). \end{aligned}$$

Indeed, a simple computation shows that $\|\tilde{M} - M\|_{L^1}$ can be bounded in terms of $\|f - \tilde{f}\|_{L^1_2}$, which in turn can be estimated in terms of $\|f_{t_1}^R\|_{L^1_2}$.

Next, the stability lemma 5.1 implies

$$\left\|f_t - \tilde{f}_t\right\| \leq C e^{C_{\text{stab}}(t-t_1)} \left\|f_{t_1} - \tilde{f}_{t_1}\right\|_{L^1_\gamma}.$$

On the whole, we find

$$\begin{aligned}\|f_t - M\|_{L^1} &\leq \left\|f_t - \tilde{f}_t\right\|_{L^1} + \left\|\tilde{f}_t - \tilde{M}\right\|_{L^1} + \left\|\tilde{M} - M\right\|_{L^1} \\ &\leq C\left(e^{C_{\text{stab}}(t-t_1)}e^{-C_{\text{dec}}t_1} + (t-t_1)^{-\frac{1}{\varepsilon}} + e^{-C_{\text{dec}}t_1}\right)\end{aligned}$$

It remains to choose $t_1 \gg (t - t_1)$ in order to compensate for the exponential divergence allowed by the stability lemma 5.1. More precisely, if $C_{\text{stab}}(t - t_1) = \frac{C_{\text{dec}}}{2}\, t_1$ (i.e $t_1 = C_{\text{stab}}/(\frac{C_{\text{dec}}}{2} + C_{\text{stab}})\, t$) then

$$\|f_t - M\|_{L^1} \leq C\left(e^{-C_{\text{stab}}(t-t_1)} + (t-t_1)^{-\frac{1}{\varepsilon}} + e^{-2C_{\text{stab}}(t-t_1)}\right)$$

and so

$$\|f_t - M\|_{L^1} \leq C(t-t_1)^{-\frac{1}{\varepsilon}} \leq C' t^{-\frac{1}{\varepsilon}}.$$

This holds for ε arbitrarily small, and the theorem is proved. □

7 Weaker integrability conditions

A natural question is wether the two main results of this paper, the decomposition theorem 5.5 and the Theorem 6.2 of convergence to equilibrium, extend to solutions with weaker integrability conditions. A first step could be $L^1_2 \cap L^p$ with $1 < p < 2$. A physically relevant assumption would be $L^1_2 \cap L \log L$. But since Mischler and Wennberg [20] have proven the existence and unicity under the sole L^1_2 assumption, the optimal assumption would be only $f_0 \in L^1_2$ (i.e no entropy condition).

It turns out that in the particular case of hard sphere collision kernel we can extend our results to general L^1_2 data, using results of [20] and [1]. A careful study of the iterated gain term $Q^+(Q^+(g,f),h)$ is done in [20] in order to prove non-concentration of the solution. This non-concentration is used to obtain the weak compactness by Dunford-Pettis Theorem and prove the existence of solution with no entropy condition. This study is refined in [1], where this iterated gain term is estimated in Lebesgue spaces. Therefore Abrahamsson is able to prove [1, Lemma 2.1]

$$\forall\ 1 \leq p < 3, \quad \|Q^+(Q^+(g,f),h)\|_{L^p} \leq C\,\|f\|_{L^1_2}\,\|g\|_{L^1_2}\,\|h\|_{L^1_2}$$

with explicit constant. He deduces a decomposition theorem [1, Proposition 2.1] from which we can extract

Lemma 7.1 (Abrahamsson's decomposition). *Let $B(v - v_*, \sigma) = |v - v_*|$, and let $f_0 \in L^1_2$ be a nonnegative initial datum with finite kinetic energy. Let f be the unique solution (with non-increasing energy) of the Boltzmann equation with collision kernel B and initial datum f. Let q be arbitrarily large and τ arbitrarily small. Then f can be decomposed as $f = f^S + f^R$ where $f^S \in L^\infty([\tau, +\infty); L^2_q \cap L^1_2)$ and for all $k \geq 0$, there is $\lambda = \lambda(k) > 0$ such that $\|f^R\|_{L^1_k} = O(e^{-\lambda t})$. All the constants in this lemma can be computed explicitely in term of the mass and energy of f_0.*

We explain how to connect this result to our method in order to get optimal assumptions on the initial data in the hard sphere case, and then we make some remarks on possible extensions for general hard potentials with cut-off.

Thus for hard spheres we have the

Theorem 7.2. *Let $B(v-v_*,\sigma) = |v-v_*|$ and $0 \leq f_0 \in L^1_2$. Let f be the unique energy-preserving solution of the Botzmann equation with initial datum f_0, and let $s \geq 0$, $q \geq 0$ be arbitrarily large. Let $\tau > 0$ be arbitrarily small. Then, for all $t \geq \tau$, f can be written $f^S + f^R$, where f^S is nonnegative, and*

$$\begin{cases} \sup_{t\geq\tau} \left\| f_t^S \right\|_{H^s_q \cap L^1_2} < +\infty \\ \\ \forall t \geq \tau,\ \forall k > 0,\ \exists \lambda = \lambda(k) > 0; \quad \left\| f_t^R \right\|_{L^1_k} = O\left(e^{-\lambda t}\right). \end{cases}$$

Moreover the conclusion of Theorem 6.1 holds true:

$$\|f_t - M\|_{L^1} = O(t^{-\infty}).$$

All the constants in this theorem can be computed in terms of τ and the mass and energy of f_0.

Proof of Theorem 7.2. First let us prove the decomposition part of the theorem. One follows the same strategy of tree decomposition as in Theorem 5.5. It is enough to take the decomposition of Lemma 7.1 at the first step of the tree: f_1 takes the smooth part of decomposition of Lemma 7.1 as initial data at time t_0. Then one has to adjust the constants in the proof: take n, the number of steps, such that $(n+1)\alpha' \geq k$ (one step more) and take

$$\mu < \frac{C_{\text{stab}}}{C_{\text{stab}} + K''}$$

where $K'' = \min\{K', \lambda\}$ (λ is the rate of exponential decrease in the decomposition of Lemma 7.1). The rest of the proof is identical to the one of Theorem 5.5. Then with the decomposition result in hand, one can prove the "almost exponential" convergence to equilibrium eactly the same way as in Theorem 6.2. $\square$

Remarks: 1. Note that except for the physically relevant case of hard spheres, the cut-off assumption is unphysical for general hard potentials interactions. Besides, non cut-off collision operators are known to have a regularizing effect (see for instance Alexandre, Desvillettes, Villani and Wennberg [2]). The optimality of the integrability condition is thus less important for general hard potentials interactions than it is for hard spheres.

2. For general hard potentials with cut-off (with $0 \leq \gamma \leq 1$), the result of Abrahamsson on the iterated gain term becomes

$$\forall\ 1 \leq p < 3, \quad \|Q^+(Q^+(g,f),h)\|_{L^p} \leq\ C_{p,\gamma}\left(\|f\|_{L^1_2}, \|g\|_{L^1_2}, \|h\|_{L^1_2}, \|f\|_{L^q}, \|g\|_{L^r}, \|b\|_{L^\infty}\right)$$

for any $1/q + 1/r < (5+\gamma)/2$. It is likely that an improvement of this result in order to allow $q = r = 1$ in this estimate would allow to extend Lemma 7.1 and thus Theorem 7.2 to general hard potentials with cut-off. However it seems that this question leads to serious technical difficulties.

3. Nevertheless a possible strategy to extend Theorem 5.5 to initial data in $L^1_2 \cap L^p$ with any $p > 1$ could be the following. In the same spirit as the tree decomposition in Theorem 5.5, one iterates the Duhamel formula, but now to increase the Lebesgue integrability at each step (using Theorem 3.6 for $s = 0$, translated into a gain of integrability thanks to the Sobolev injections coupled with some interpolation). As soon as the L^2 integrability is reached, one can start the decomposition tree of Theorem 5.5 in order to increase regularity, connecting the two decompositions in the same spirit as in the proof of Theorem 7.2.

Appendix: Some facts from interpolation theory and harmonic analysis

The goal of this appendix is to recall some classical results about linear interpolation theory and also to give the proof of some elementary results used here, in order to make this paper almost self-contained.

Convolution inequalities in weighted spaces

Proposition A.1. *Let $\eta \in \mathbb{R}$, then*

$$\|f * g\|_{L^r_\eta} \le \|f\|_{L^p_{|\eta|}} \|g\|_{L^q_\eta}$$

for all $p, q, r \ge 1$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

The proof of this proposition is exactly similar to the standard proof of the usual Young inequality.

Riesz-Thorin interpolation

Proposition A.2. *Let $\theta \in [0,1]$, $p_1, p_2, p \in [1,+\infty]$ such that $1/p = \theta/p_1 + (1-\theta)/p_2$, $k_1, k_2, k \in \mathbb{R}$ such that $k = \theta k_1 + (1-\theta)k_2$, $q_1, q_2, q \in [1,+\infty]$ such that $1/q = \theta/q_1 + (1-\theta)/q_2$, $l_1, l_2, l \in \mathbb{R}$ such that $l = \theta l_1 + (1-\theta)l_2$, and let T be a continuous operator from $L^{p_1}_{k_1}$ into $L^{q_1}_{l_1}$ and from $L^{p_2}_{k_2}$ into $L^{q_2}_{l_2}$. Then its restrictions to C_0^∞ functions extends to a continuous operator from L^p_k into L^q_l with the following bound on its norm*

$$\|T\|_{L^p_k \to L^q_l} \le \|T\|^{\theta}_{L^{p_1}_{k_1} \to L^{q_1}_{l_1}} \|T\|^{1-\theta}_{L^{p_2}_{k_2} \to L^{q_2}_{l_2}}.$$

Corollary A.3. *Let $\theta \in [0,1]$, $s_1, s_2, s \in \mathbb{R}$ such that $s = \theta s_1 + (1-\theta)s_2$, and $t_1, t_2, t \in \mathbb{R}$ such that $t = \theta t_1 + (1-\theta)t_2$. If T is a continuous operator from H^{s_1} into H^{t_1} and from H^{s_2} into H^{t_2}, then its restriction to C_0^∞ functions extends to a continuous operator from H^s into H^t with the following bound on its norm*

$$\|T\|_{H^s \to H^t} \leq \|T\|^{\theta}_{H^{s_1} \to H^{t_1}} \|T\|^{1-\theta}_{H^{s_2} \to H^{t_2}} .$$

This corollary is still true when one adds a weight on the space variable:

$$\|T\|_{H^s_k \to H^t_{k'}} \leq \|T\|^{\theta}_{H^{s_1}_k \to H^{t_1}_{k'}} \|T\|^{1-\theta}_{H^{s_2}_k \to H^{t_2}_{k'}} .$$

In fact the abstract method of interpolation leads to the stronger result

$$\|T\|_{H^s_k \to H^t_{k'}} \leq \|T\|^{\theta}_{H^{s_1}_{k_1} \to H^{t_1}_{k'_1}} \|T\|^{1-\theta}_{H^{s_2}_{k_2} \to H^{t_2}_{k'_2}}$$

where the weight indexes satisfy $k = \theta k_1 + (1-\theta)k_2$ and $k' = \theta k'_1 + (1-\theta)k'_2$. As a consequence one could prove a strong version of the Young inequality in the case of the weighted Sobolev spaces. One can indeed make the index of weight and regularity vary together. Namely

$$\|f\|_{H^s_k} \leq \|f\|^{\theta}_{H^{s_1}_{k_1}} \|f\|^{1-\theta}_{H^{s_2}_{k_2}}$$

where $s = \theta s_1 + (1-\theta)s_2$ and $k = \theta k_1 + (1-\theta)k_2$. Let us emphasize the consequence of this inequality that we use in this paper: as soon as f belongs to H^s and has finite L^1 moments of order large enough, one can deduce bounds on $H^{s'}_k$ norm for $s' < s$.

Regularity of a sum $H^s_k + H^{s+\beta}_k$

Theorem A.4. *Let $h \in H^s_\eta$ ($s \geq 0$, $\eta \in \mathbb{R}$) such that for all ε small enough,*

$$h = h_1^\varepsilon + h_2^\varepsilon$$

where the two parts h_1^ε and h_2^ε satisfy the following estimates: there exist $k_1 \geq 0$ and $k_2 > 0$ such that $\|h_1^\varepsilon\|_{H^{s+\beta}_\eta} \leq C_1 \varepsilon^{-k_1}$ et $\|h_2^\varepsilon\|_{H^s_\eta} \leq C_2 \varepsilon^{k_2}$ ($\beta > 0$). Then

$$h \in H^{s+\alpha}_\eta, \quad \forall \alpha < \frac{\beta k_2}{k_1 + k_2}.$$

Remarks: 1.Our estimate on the norm $H^{s+\alpha}_\eta$ blows up like $\frac{\mathrm{cst}}{\frac{\beta k_2}{k_1+k_2} - \alpha}$ as $\alpha \to \frac{\beta k_2}{k_1+k_2}$.

2. In fact the proof shows that $h\langle \cdot \rangle^\eta$ belongs to the Besov space $B^{\infty,\infty}_\beta \subset \cap_{\alpha<\beta} H_\alpha$.

Proof of theorem A.4. Let us take $\alpha < \frac{\beta k_2}{k_1+k_2}$. Without loss of generality we treat the case $s = \eta = 0$ (the general case can be reduced to this one). We first prove an upper bound on an annulus. Let $0 < A < B$, and

$$\begin{aligned}
\int_{A\leq|\xi|\leq B} |\widehat{h}(\xi)|^2\langle\xi\rangle^{2\alpha}d\xi &\leq 2\int_{A\leq|\xi|\leq B}\left(|\widehat{h_1^\varepsilon}(\xi)|^2 + |\widehat{h_2^\varepsilon}(\xi)|^2\right)\langle\xi\rangle^{2\alpha}d\xi \\
&\leq 2\left(C_1\varepsilon^{-k_1}\langle A\rangle^{2(\alpha-\beta)} + C_2\langle B\rangle^{2\alpha}\varepsilon^{k_2}\right) \\
&\leq 2\left(C_1\varepsilon^{-k_1}A^{2(\alpha-\beta)} + 2C_2B^{2\alpha}\varepsilon^{k_2}\right) \\
&\leq \max(2C_1, 4C_2)\left(\varepsilon^{-k_1}A^{2(\alpha-\beta)} + B^{2\alpha}\varepsilon^{k_2}\right)
\end{aligned}$$

As this inequality holds for all ε, one can choose it in order that the two right-members be equal in the preceding inequality. The computation leads to

$$\int_{A\leq|\xi|\leq B} |\widehat{h}(\xi)|^2\langle\xi\rangle^{\beta}d\xi \leq 2\max(2C_1, 4C_2)B^{\frac{2\alpha k_1}{k_1+k_2}}A^{\frac{2(\alpha-\beta)k_2}{k_1+k_2}}$$

Let $C_3 = 2\max(2C_1, 4C_2)$ and let us sum the inequalities on a family of concentric dyadic annuli:

$$\begin{aligned}
\|h\|_{H^\beta} &\leq \int_{0\leq|\xi|\leq 1} |\widehat{h}(\xi)|^2\langle\xi\rangle^{\beta} + C_3\sum_{n=0}^{+\infty} 2^{\frac{2\alpha(n+1)k_1}{k_1+k_2}}2^{\frac{2(\alpha-\beta)nk_2}{k_1+k_2}} \\
&\leq 2\|h\|_{L^2} + C_3\sum_{n=0}^{+\infty} 2^{\frac{2\alpha(n+1)k_1}{k_1+k_2}}2^{\frac{2(\alpha-\beta)nk_2}{k_1+k_2}} \\
&\leq 2\|h\|_{L^2} + C_3 4^{\frac{\alpha k_1}{k_1+k_2}}\sum_{n=0}^{+\infty} 4^{n(\alpha-\frac{\beta k_2}{k_1+k_2})}.
\end{aligned}$$

Thanks to the assumption on α the right member is summable and thus $h \in H^\alpha$ with the following bound on the norm

$$\begin{aligned}
\|h\|_{H^\alpha} &\leq 2\|h\|_{L^2} + C_3 4^{\frac{\alpha k_1}{k_1+k_2}}\sum_{n=0}^{+\infty} 4^{n(\alpha-\frac{\beta k_2}{k_1+k_2})} \\
&\leq 2\|h\|_{L^2} + C_3\frac{4^{\frac{\alpha k_1}{k_1+k_2}}}{1-4^{\alpha-\frac{\beta k_2}{k_1+k_2}}}.
\end{aligned}$$

$\square$

A simple estimate on pseudo-differential operators

We conclude this appendix with a simple result needed for the proof of the regularity property of Q^+. This can be linked with more general pseudo-differential estimates, but will be proved by elementary means. The space H^s_k is not an algebra in general (it is an algebra thanks to the Sobolev imbeddings as soon as $2s > N$), but one can prove a bound on the norm H^s_k of a product of functions if one of the two functions has regularity greater than s:

$$\|fg\|_{H^s_k} \leq \mathrm{cst}(N,\varepsilon)\ \|f\|_{H^S_{k_1}}\ \|g\|_{H^s_{k_2}}$$

where $k_1 + k_2 = k$, and $S = s + N/2 + \varepsilon$ with $\varepsilon > 0$.

Now we follow the same idea but assuming that one of the two functions depends also on the Fourier variable.

Lemma A.5. *Let $\psi(x,\xi)$ be a real-valued C^∞ function on $\mathbb{R}^N \times \mathbb{R}^N$, compactly supported in x, uniformly in ξ. Let $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, and let $s \in \mathbb{R}$. Let us define*

$$I = \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} \Big|\mathcal{F}(g(\cdot)\psi(\cdot,\xi))\Big|^2 d\xi.$$

Then for all $\varepsilon > 0$ there exists a constant $\mathrm{cst}(N,\varepsilon)$ such that

$$I \leq \mathrm{cst}(N,\varepsilon)\, \|\psi\|^2_{L^\infty_\xi(H^S_x)}\, \|g\|^2_{H^s},$$

with $S = s + N/2 + \varepsilon$.

Proof of Lemma A.5. We have

$$g(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot\tau}\hat{g}(\tau)d\tau$$

hence

$$\begin{aligned}
I &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\langle\xi\rangle^{2s}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ix\cdot(\xi-\tau)}\hat{g}(\tau)\psi(x,\xi)dxd\tau\right|^2 d\xi \\
&= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\langle\xi\rangle^{2s}\left|\int_{\mathbb{R}^N}\hat{g}(\tau)\left[\int_{\mathbb{R}^N} e^{-ix\cdot(\xi-\tau)}\psi(x,\xi)dx\right]d\tau\right|^2 d\xi \\
&= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\langle\xi\rangle^{2s}\left|\int_{\mathbb{R}^N}\hat{g}(\xi-\tau)\left[\int_{\mathbb{R}^N} e^{-ix\cdot\tau}\psi(x,\xi)dx\right]d\tau\right|^2 d\xi \\
&= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\langle\xi\rangle^{2s}\left|\int_{\mathbb{R}^N}\hat{g}(\xi-\tau)\left[\mathcal{F}_x(\psi(\cdot,\xi))(\tau)\right]d\tau\right|^2 d\xi
\end{aligned}$$

and thus

$$\begin{aligned}
(2\pi)^N I &\le \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} \int_{\mathbb{R}^N} |\hat{g}|^2(\xi-\tau)\langle\tau\rangle^{-2S} d\tau \\
&\qquad \int_{\mathbb{R}^N} \left|\mathcal{F}_x(\psi(\cdot,\xi))\right|^2(\tau')\langle\tau'\rangle^{2S} d\tau' d\xi \\
&\le \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} \int_{\mathbb{R}^N} |\hat{g}|^2(\xi-\tau)\langle\tau\rangle^{-2S} d\tau \, \|\psi(\cdot,\xi)\|^2_{H^S_x} d\xi \\
&\le \|\psi\|^2_{L^\infty_\xi(H^S_x)} \int_{\mathbb{R}^N} \langle\tau\rangle^{-2S} \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} |\hat{g}|^2(\xi-\tau) d\xi \, d\tau \\
&\le \|\psi\|^2_{L^\infty_\xi(H^S_x)} \int_{\mathbb{R}^N} \langle\tau\rangle^{-2S} \int_{\mathbb{R}^N} \langle\xi+\tau\rangle^{2s} |\hat{g}(\xi)|^2 d\xi \, d\tau \\
&\le \|\psi\|^2_{L^\infty_\xi(H^S_x)} \, 2^s \int_{\mathbb{R}^N} \langle\xi\rangle^{2s} \hat{g}(\xi)^2 d\xi \int_{\mathbb{R}^N} \langle\tau\rangle^{-2S} \langle\tau\rangle^{2s} d\tau \\
&\le \|\psi\|^2_{L^\infty_\xi(H^S_x)} \, \|g\|^2_{H^s} \, 2^s \int_{\mathbb{R}^N} \langle\tau\rangle^{-N-2\varepsilon} d\tau
\end{aligned}$$

which concludes the proof. □

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